

Arvid C. Johnson, Graduate School of Business and Information Systems Dominican University,
7900 W. Division St., River Forest, IL 60305 Phone: 708-524-6465 - Fax: 708-524-6939 -
ajohnson@email.dom.edu

Nick T. Thomopoulos, Stuart Graduate School of Business Illinois Institute of Technology, 565 W.
Adams St., Chicago, IL 60661 Phone: 312-906-6536 - Fax: 312-936-6549 - thomop@stuart.iit.edu

Proceedings of POM High Tech, 2002, 18 p.

Abstract

Truncated normal distributions have found utility in a wide variety of fields – including those within the purview of production and operations management. Despite this fact, work to date has focused primarily on estimation of the original (non-truncated) population parameters based upon truncated or censored samples rather than on the characteristics of the truncated distributions themselves. This paper presents selected reference tables of the cumulative distribution function of the doubly-truncated normal distribution for use by workers/researchers – developed using Microsoft Excel's Visual Basic for Applications environment. In the process of developing these tables, the characteristic parameters of these distributions are derived in terms of the standard normal distribution. In addition, it is noted that the points of truncation do not uniquely determine the value of the coefficient of variation.

Key words: Probability Models Truncated Normal Distribution Cumulative Distribution Function

The normal – or Gaussian – distribution is one of the most widely utilized of all random variables. Mound-shaped or approximately mound-shaped distributions are encountered in a large number of applications and, via the Central Limit Theorem, provide the underpinning for the characteristics of sampling distributions upon which statistical inference is based. Despite this utility, the fact that the values of a normally distributed random variable can, in theory, assume any value over the range from $-\infty$ to $+\infty$ may lead to significant computational errors in situations in which the distribution's outcomes are constrained. This problem has motivated the study of truncated normal distributions.

The study of the properties of normally distributed random variables when certain outcomes are constrained or restricted has been a rich and fertile one – with applications reported in the areas of inventory management, regression analysis, and financial modeling to name only a few. Indeed, the characteristic parameters (i.e., mean μ and standard deviation σ) of truncated normal distributions can be readily derived using basic statistical methods. The vast majority of the work to date has focused on estimating the parameters of the original, non-truncated population based upon the sample statistics of the truncated or censored samples. Little work has been done in terms of describing and tabulating the properties of the truncated normal distributions themselves.

This paper presents tables of the cumulative distribution function of the doubly-truncated normal distribution for use by workers/researchers in production and operations management, finance, and the many other fields in which this family of distributions has found application. In the process of developing these tables, the characteristic parameters of these distributions are derived in terms of the standard normal distribution – for maximum generality. After a brief review of prior research in this field, this paper presents a detailed development of the

Review of Prior Research

While several, different notations – often incorporating intermediately-defined functions for convenience – have been used in the literature, the notational conventions presented in [Hald, 1952], which describe truncated normal distributions in terms of the probability density and cumulative distribution functions of the non-truncated normal distribution, will be used in this paper.

There is a large body of literature on the subject of the estimation of the parameters of the original population (μ , σ) based upon data from truncated or censored samples. [Schneider, 1986] provides an excellent overview of parameter estimation of truncated normal distributions in his Chapter 2. One of the founders – and most prolific authors – in this field is A. Clifford Cohen at the University of Georgia, whose works span over four decades. (See, for example [Cohen, 1991].) In general, the procedures/algorithms that have been developed depend upon whether the points of truncation are known. The focus of this paper – like that of [Barr and Sherrill, 1999] – is on the characteristics of the truncated population rather than the original, non-truncated population.

Little work has been done in terms of generating tables – especially cumulative probability tables – and other representations for the truncated normal distribution. [Hald, 1952] presents a probability-paper graph of the cumulative distribution functions of three left-truncated normal distributions with degrees of truncation of 10%, 30%, and 50%. [Kececioglu, 1991] displays the same three distributions. [Johnson and Kotz, 1970] provides qualitative illustrations of the probability density functions associated with the nine combinations of truncation that result from 0%, 20%, and 40% truncation on the left/lower and right/upper sides of the deviation divided by the standard deviation) of the truncated normal distribution for 23 selected combinations of right- and left-truncation ranging from 0 to 50%. [Thomopoulos, 1980] has qualitative visualizations of the left-truncated normal distribution for four truncation points and a tabular summary of the mean, standard deviation, and coefficient of variation of the left-truncated normal distribution for truncation points over a range similar to that discussed in this paper. [Barr and Sherrill, 1999] plots the mean and variance of the left-truncated normal distribution as a function of the truncation point (degree of truncation).

Doubly-truncated Normal Distribution

Definition:

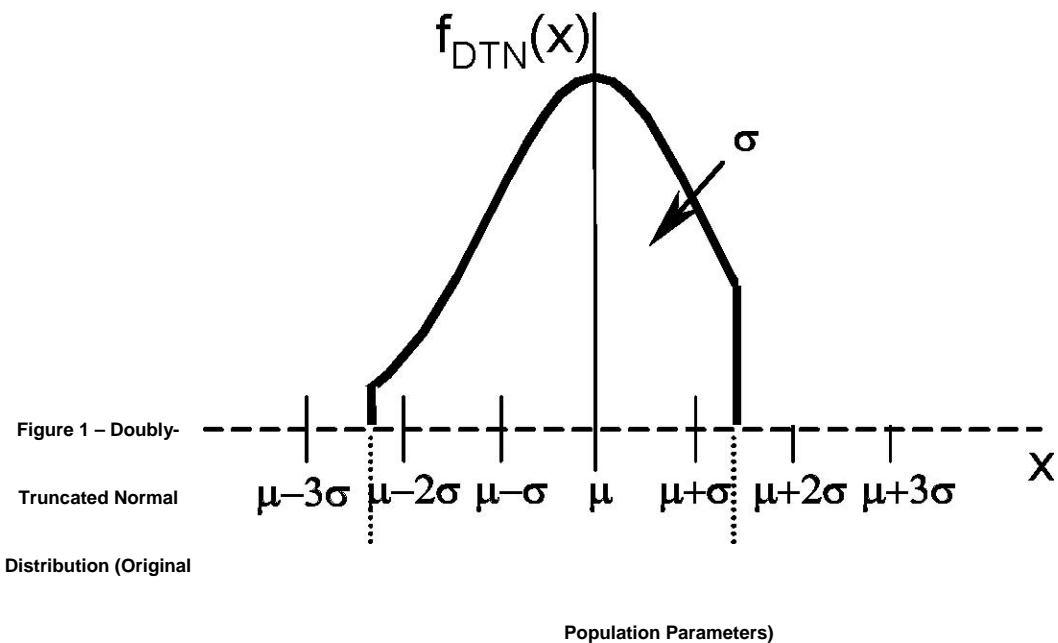
Consider a normally-distributed random variable x with a probability density function $f(x)$ specified as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If the values of x below some value x_L and above some value x_R cannot be observed – due to censoring or truncation – then, as shown in Figure 1 and following [Hald, 1952]’s conventions, the resulting distribution is a doubly-truncated normal distribution with probability density function $f_{DTN}(x)$ given by

$$f_{DTN}(x) = \begin{cases} 0, & -\infty \leq x \leq x_L \\ \frac{\sqrt{2\pi\sigma^2}}{\int_{x_L}^{x_R} f(x) dx}, & x_L \leq x \leq x_R \\ 0, & x \leq -\infty \text{ or } x \geq \infty \end{cases}$$

where $f(x)$ is as defined in Equation 1.



It is worth noting at this point that, while Equation 2 describes the case of the doubly-truncated normal distribution, the equation can be modified for the left-truncated distribution by allowing the upper truncation point x_R to approach $+\infty$. Similarly, a right-truncated normal distribution results from allowing the lower truncation point x_L to approach $-\infty$.

For purposes of generality, Equation 2 can be re-stated in terms of the standard normal distribution (denoted $f(z)$ – with a mean of 0 and a standard deviation of 1) where

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty \leq z \leq \infty \quad (3)$$

$$z = \frac{x - \mu}{\sigma} \quad (4)$$

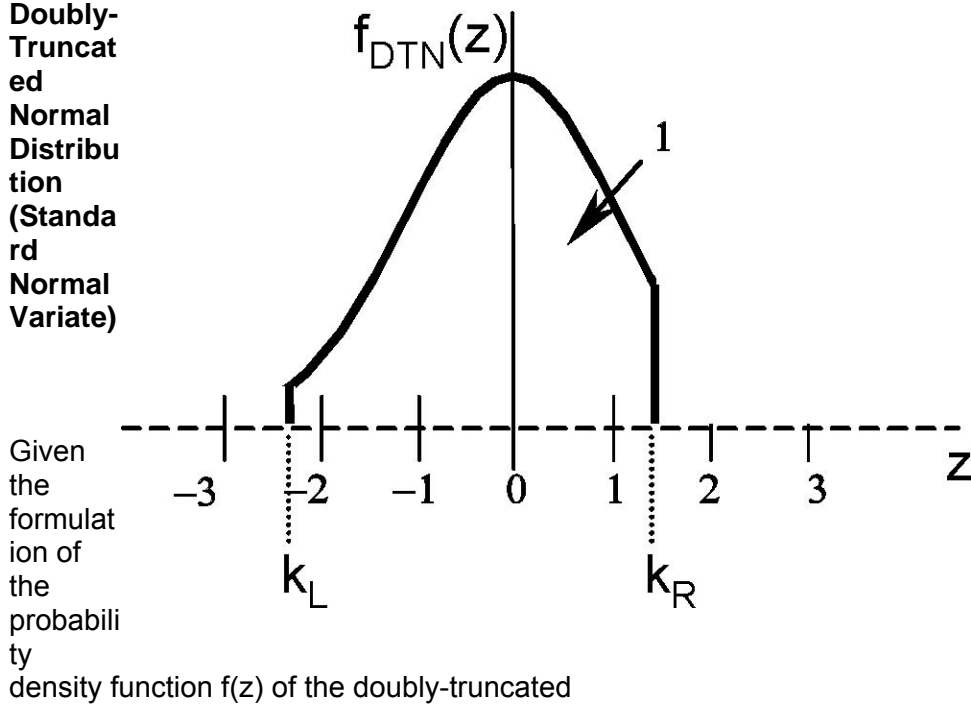
In terms of this standard normal distribution, the points of truncation x_L and x_R will be denoted by k_L and k_R , respectively, as given by

$$x_L - \mu = \sigma k_L \quad \text{and} \quad x_R - \mu = \sigma k_R \quad (5)$$

$$f_{DTN}(z) = \begin{cases} 0, & -\infty \leq z \leq k_L \\ \frac{f(z) - F(k_L)}{F(k_R) - F(k_L)}, & k_L \leq z \leq k_R \\ 0, & k_R \leq z \leq \infty \end{cases}$$

This is illustrated in Figure 2.

Figure 2 – Doubly-Truncated Normal Distribution (Standard Normal Variate)



normal distribution in Equation 6, its cumulative distribution function $F(z)$ can be stated as

$$F(z) = \begin{cases} 0, & z \leq k_L \\ \frac{F(z) - F(k_L)}{F(k_R) - F(k_L)}, & k_L \leq z \leq k_R \\ 1, & z \geq k_R \end{cases} \quad (7)$$

$$\int_{-\infty}^z$$

Equations 6 and 7 – for the probability density and cumulative probability functions of the truncated normal distribution, respectively – are equivalent to those presented in [Hald, 1952], [Schneider, 1986], [Cohen, 1991], and [Johnson and Kotz, 1970].

Some authors, rather than specifying the truncation points k_L and k_R , refer to the “degree of truncation” – i.e., the fraction of the non-truncated distribution that cannot be observed. On the lower/left side, the degree of truncation is given by $F(k_L)$ and, on the upper/right side, by $1 - F(k_R)$, as noted in [Johnson and Kotz, 1970]. From this, the degree of truncation of a doubly-truncated normal distribution can easily be seen to be $1 - [F(k_R) - F(k_L)]$. In fact, [Kececioglu, 1991] uses the degree of truncation – expressed as a fraction of the original population – to motivate and explain Equation 7.

Parameters:

In this section, the parameters of the truncated normal distribution are discussed – given points of truncation x_L and x_R . Specifically, expressions for the mean and variance of the truncated normal distribution are addressed – adopting the convention of describing the mean ($\mu = E(z)$) and variance ($\sigma^2 = V(z)$) of the truncated normal distribution in terms of the standard normal variate z . Given $E(z)$ and $V(z)$, the mean and variance of the truncated normal distribution in terms of the original population parameters can be found using

$$V(x) = \sigma^2 V(z) \quad (9)$$

and

$$E(x) = \mu + \sigma E(z) \quad (10)$$

probability density $f(z)$ and cumulative probability $F(z)$ functions of the standard normal distribution. Their expressions for the doubly-truncated normal distribution – in terms of the notation utilized in this paper – are shown in Equations 11 and 12.

$$E(z) = \frac{\int_{k_L}^{k_R} z f(z) dz}{F(k_R) - F(k_L)} \quad (11)$$

$$F(z) = \frac{\int_{k_L}^z f(t) dt}{F(k_R) - F(k_L)} \quad (12)$$

Since Equations 11 and 12 are expressed entirely in terms of $f(z)$ and $F(z)$, they may be readily implemented using spreadsheet software, such as Microsoft Excel, or via approximations for these functions such as those given in [Abramowitz and Stegun, 1972].

From Equations 11 and 12, it is clear that the mean and standard deviation of the doubly-truncated normal distribution are solely dependent upon the points of truncation k_L and k_R . The coefficient of variation c – which, again, is determined solely by the particular values of k_L and k_R – can be defined as

$$c = \frac{\sigma_z}{\mu_z} \quad (13)$$

Development of Tables:

With the formulation of the cumulative distribution function $F_{DTN}(z)$ of the doubly-truncated normal distribution presented in Equation 7, given points of truncation k_L and k_R , one can readily calculate the value of the cumulative distribution function of the doubly-truncated normal distribution at any value of z through the use of tables of the cumulative distribution function of the standard normal distribution $F(z)$.

With the wide availability of desktop micro-computers and their associated software, alternate methods of evaluating Equation 7 at specific values of z include the use of “standard” spreadsheet functions (e.g., NormSDist(Z)) within Microsoft Excel), subroutines and functions within Microsoft Excel's Visual Basic for Applications (VBA) environment, or formulations such as those provided in [Abramowitz and Stegun, 1972]. The Excel-based VBA approach has been utilized in preparing the tables presented in this paper due to its ease of implementation.

Using the approach outlined above, a table of the standard normal variate z at which the cumulative distribution function $F_{DTN}(z)$ of the doubly-truncated normal distribution function assumes some value has been developed. This problem was solved computationally through the use of a dichotomous line search algorithm, and the results are presented in Tables 1 through 6 for allowable combinations of k_L and k_R over the range $-3.0(0.2)3.0$. (“Allowable combinations” simply means that the upper truncation point k_R must be greater than the lower truncation point k_L .) Tables 1 through 6 are presented in a format similar to the tables often provided for functions such as the “F” distribution. In this case, for six values of the cumulative distribution function $F_{DTN}(z)$, Tables 1 through 6 provide the standard z value associated with that $F_{DTN}(z)$ as a function of the points of truncation k_L and k_R – for $F_{DTN}(z) = 0.01, 0.025, 0.05, 0.95, 0.975, \text{ and } 0.99$, respectively.

Unlike the case of singly-truncated normal distributions (See [Johnson, 2001].), the coefficient of variation c is not uniquely determined by the points of truncation k_L and k_R for the doubly-truncated normal distribution. This is illustrated in Figure 3, which presents a contour plot of the magnitude of the coefficient of variation over the range of allowable truncation points from -3 to 3 . In reviewing Figure 3, several points are worth noting:

- The figure possesses the symmetry that would be expected.
- The “sign” of the coefficient of variation is determined by the relative degree of truncation of the lower and upper tails – which, in turn, determines the sign of the mean.
- The coefficient of variation c approaches $\pm\infty$ for cases of symmetric truncation and becomes undefined.

0 2.5
93
3
-
2.5
86
9-
2.5
77
6-
2.5
64
7-
2.5
47
7-
2.5
26
2

-
2.3
97
5

-
2.5
97
4-

-		5
2		3
.		-
7		2
5		.
6		2
3		7
		1
-		4
2		-
.		2
7		.
0		2
1		4
2		6
		9
-		-
2		2
.		1
6	-	6
4	2	0
5	.	9
5	5	0
	0	0
-	0	9
2	5	-
.		2
5	-	.
9	2	1
1	.	4
2	4	7
	7	7
-	1	3
2	2	-
.		2
5	-	.
3	2	1
9	.	3
9	4	3
	3	1
-	9	9
2	3	-
.		2
6	-	.
4	2	1
2	.	1
4	4	1
	0	5
-	6	1
2	2	-
.		2
6	-	.
0	2	0
1	.	9
0	3	7
	7	7
-	3	6
2	0	-
.		1
5		.
5		9
7		7
4		7
	-	5
-	2	4
2	.	-
.	3	1
5	3	.
1	7	9
3	7	6
6		6
	-	1
-	2	-
2	.	1
.	3	.
4	1	9
7	7	5
1	6	5
0		4
	-	-
	2	-
	.	-
	2	-
	9	1

1.2
1.4
1.6
1.8
.
2.2
2.4
2.6
2.8
3
-3

2

-
2
.
4
9
2
8
-
2
.
4
5
0
7
-
2
.
2.8
1
3
9
-
2
.
3
8
2
7
-
2
.
3
5
6
7

V
a
l
u
e
s

n
o
t

a
l
l
o
w
e
d

-2.4309 -2.3945 -2.3622 -2.3344 -2.3112
-2.2923 -2.2775 -2.2661 -2.2577 -2.2517

-
1
.
7
9
7
6
-
1
.
7
9
4
3
-
1
.
7
9
0
1

.
9
4
3
7
-
1
.
9
3
1
2

6
-
2
.
2
8
1
7
-
2
.
2
8
0
5
-
2
.
2
7
9
8

-
2
.
2
4
7
6
-
2
.
2
4
4
8
-
2
.
2
4
3
0
-
2
.
2
4
1
9

-
2
.
2
8
6
6

-
2
.
2
8
3

-
1
.
1
8
0
4

-
1.5
97
7-
1.5
94
6
-
1.3
97
7-
-
1.2

2.3
35
8-
2.3
19
4-
2.3
06
9-
2.2
97
7-
2.2
91
2

0

0.2
0.4
0.6
0.8
.

1

-
2.6

-
2
.3
4
1
1
-
2
.3
1
1
3
-
2
.2
2
2.4
4
6
-
2
.2
6
1
2
-
2
.2
2
2
2
4
1
5

-
2
.2
2
2
8
-2.2
2

-
2.2
25
4-
2.2
12
6-
2.2
02
7-
2.1
95
4-
2.1
90
2

.1
9
9
-
2
.1
7
8
9
-
2
.1
6
0
3
-
2
.1
4
4
5

-2.0800 -2.0630 -2.0472 -2.0331 -2.0209
-2.0107 -2.0025 -1.9962 -1.9914 -1.9880

2
.1
8
1
6
-
2
.1
8
1
1

-
2
.0
9
9
9
5
-
2
.0
9
7
5
-
2
.0
9
6
2
-
2
.0
9
5
4
-
2
.0
9
4
9

-
2
.1
8
6
6

-
2
.1
8
4
1

-
2
.1
8
2
6

-
1
.9
8
5
6
-
1
.9
8
3
0
-
1
.9
8
2
4
-
-

979 -1.6952 -1.6932

1
.9
8
2
0

6
9

-
1
.
6
9
1
8
4
6
1
-
1
.
8
4
5
6
-
1
.
6
9
0
9
-
1
.
6
8
9
9
-
1
.
6
8
9
7

-
1
.
8
4
5
1

-2

-
1
.
9
1
8
5
-
1
.
9
0.8
6
1

-
1
.
8
9
4
5

-
1
.
8
8
4
0
-
1
.
7
3
5
8
7
4
9

-
1.8
67
2-
1.8
61
0-
1.8
56
2-
1.8
52
5-
1.8
49
9

-
1.7
06
2-
1.7
01
5-
1.6

-
1
.
7
4
1
-
1
.
7
2
6
5
-
1
.
7
1
8
8
-
1
.
7
1
2
0

-1.5611 -1.5544 -1.5481 -1.5422 -1.5371
-1.5328 -1.5293 -1.5265 -1.5244 -1.5229

-
1
.
8
4
8
1
-
1
.
8
4

-
1
.
5
2
1
8
-
1
.
5
2
1
1
-
1
.
5
2
0
7
-
1
.
5
2
0
2

3-1.1574

1.4

-
 1
 .
 3
 7
 2
 5
 -
 1
 .
 3
 6
 7
 4
 -
 1
 .
 3
 6
 2
 6
 -
 1
 .
 8
 0
 4
 5
 8
 2
 -
 1
 .
 7
 6
 4
 4
 2
 -1
 1
 .
 1
 3
 50
 9-
 1.3
 48
 2-
 1.3
 46
 1-
 1.3
 44
 5-
 1.3
 43
 3
 -
 1.1
 63
 4-
 1.1
 61
 3-
 1.1
 59
 6-
 1.1
 58

-
 1
 .
 8
 0
 4
 -
 1
 .
 7
 6
 4
 -1
 1
 .
 7
 2
 6
 -
 1
 .
 6
 6
 0

-0.9860 -0.9828 -0.9797 -0.9768 -0.9743
 -0.9722 -0.9704 -0.9690 -0.9680 -0.9673

.
 3
 4
 1
 6
 -
 1
 .
 3
 4
 1
 4
 -
 1
 .
 3
 4
 1
 2
 -
 1
 .
 5
 6
 0
 -
 1
 .
 1
 .
 5
 5
 8
 -
 1
 .
 1
 .
 5
 5
 7
 -
 1
 .
 0
 .
 9
 6
 6
 7
 -
 1
 .
 3
 4
 2
 5
 -
 1
 .
 3
 4
 1
 9
 -
 1

-
 1
 .
 5
 2
 0
 1

-
 1
 .
 3
 4
 1
 2

-0.5831 -0.5818 -0.5808 -0.5800 -0.5794

0 7
. 3
9 9
6 -
6 0
0 .
- 7
0 7
. 3
9 6
6 -
5 0
9 .

-
0
. 5
7
9
0
-
0
. 5
7
8
8
-
0
. 5
7
8
6
-
0
. 5
7
8
5
-
0
. 5
7
8
4

-
0
. 7
7
3
1

-
0.8

-
0
. 7
9
0
1
-
0
. 7
0.6
7
4

-0.4

-
0
. 7
8
4
8
-
0
. 5
9
3
7
2
8
2
4
-
0
. 7
8
0
3

-0.3958 -0.3936 -0.3916 -0.3897 -0.3880
-0.3865 -0.3854 -0.3844 -0.3837 -0.3832

-
0.7
78
5-
0.7
77
0-
0.7
75
8-
0.7
74
9-
0.7
74
3

. 5
8
8
6
-
0
. 5
8
6
5
-
0
. 5
8
4
6

-
0
. 7
-
0
. 3
8
2
4

-
0
. 5
7
8
4

-
0.2

-
0
.
1
9
8
0
0
.
10
9
5
9

0.2

0
.
0
0
2
0
0
.
0
0
3
9
0
.
0
0
5
7
0
.
0
0
7
2

0
.
2
0
1
9
0
.
2
0
3
7
0
.
2
0
5
3

0.2067 0.2078 0.2087 0.2094 0.2098

-
0.1
89
3-
0.1
88
1-
0.1
87
3-
0.1
86
6-
0.1
86
1
0.0
08
6
0.0
09
7
0.0
10
5
0.0
11
2
0.0
11
6

-
0
.
3
8
2
3

1
8
5
6
-
0
.
1
8
5
4
-
0
.
1
8
5
3
-
0
.
1
8
5
3

1
2
2
0
.
0
1
2
3
0
.
0
1
2
4
0
.
0
1
2
5

-
0
.
1
8
5
2

0
.
2
1
0
2
0
.
2
1
0
4
0
.
2
1
0
6
0
.
0
1
2
1
0
7
0

-
0
.
1
8
5
8
-
0
.
1
8
5
8

0
.
0
1
2
0
0
.
0
0

03
5
0.6
04
8
0.6
05
8
0.6
06
6
0.6
07
2

0.4
0.4
01
9
0.4
03
6

0.4
05
1
0.4
06
2
0.4
07
2
0.4
07
9
0.4
08
4

1

1.0018 1.0032 1.0043 1.0051

0.8
0.8
01
8
0.8
03
3
0.8
04
5
0.8
05
4
0.8
06
1

1.2

0.6
0.6
01
9
0.6

.
2
1
0
7

0
.8
0
7
1

0
.6
0
7
6

0
.6
0
7
8

0
.6
0
8
0

0
.6
0
8
1

0
.6
0
8
2

0
.4
0
9
2

0
.4
0
9
2

0
.4
0
9
3

0
.8
0
6
5

0
.8
0
6
9

0
.2
1
0
7

0
.4
0
9
3

0
.6
0
8
2

1.8

0
3
1
1
.
2
0
4
1

0
3
9
1
.
4
0
4
5

1
.
8
0
1
7
1
.
8
0
2
8
1
.
8
0
3
5
1
.
8
0
4
0
1
.
8
0
4
2

1
.
2
0
4
8
1
.
2
0
5
2
1
.
2
0
5
5
1
.
2
0
5
7

1
.
4
0
4
9
1
.
4
0
5
1
1
.
4
0
5
2

1
.
2
0
5
8

1
.
6
0
2
9

2
.
0
0
1
6
2
.
0
0
2
7
2
.
0
0
3
4
2
.
0
0
3
8

1
.
2
0
5
9

1.6

2

1.4

V
a
l
u
e
s

n
o
t

a
l
l
o
w
e
d

1.4
01
7
1.4
03
0

2.2

1.6
01
7

1
.
4

2
.
2
0
1

2.4
2.4
01
6
2.4
02
5
2.6
2.6016
2.8

6
2
.2
0
2
6
2
.2
0
3
2

2
.2
0
3
5
2
.

0 2.5
83
4
-
2.5
68
1 -
2.5
46
3 -
2.5
17
2 -
2.4
80
3 -
2.4
36
2

-
2.3
93
7

-
2.5
93
5 -

-		4
2		1
.		-
5		2
6		1
8		2
		9
-		5
2		-
.		2
4		7
7		0
6		8
		5
-		5
2		-
.		2
4	-	2
0	2	.
1	.	1
1	3	0
	8	7
-	6	8
2	2	-
.		2
3	-	.
2	2	0
9	.	7
0	3	8
	3	1
-	2	
2	3	-
.		2
2	-	.
6	2	0
2	.	4
5	2	5
	7	3
-	6	
2	7	-
.		2
4	-	.
8	2	0
6	.	1
1	2	0
	2	9
-	1	
2	5	-
.		1
4	-	.
1	2	9
8	.	7
7	1	6
	6	1
-	8	
2	4	-
.		1
3		.
5		9
1		4
5		0
	-	7
-	2	
.	.	-
2	2	1
8	5	.
6	8	9
7	4	1
		9
-	-	2
2	2	-
.	2	1
2	1	.
2	7	8
5	7	9
9		5
	-	1
	2	
	.	-
	1	1
	7	

1.2
1.4
1.6
1.8
•

2.2
2.4
2.6
2.8

3
-3

2

-
2
.
2
0
2
5

-
2
.
1
4
9
5

-
2
.
2.8
0
3
7

-
2
.
0
6
5
0

-
2
.
0
3
3
1

V
a
l
u
e
s

n
o
t

a
l
l
o
w
e
d

-2.1703 -2.1207 -2.0775 -2.0408 -2.0104
-1.9858 -1.9666 -1.9519 -1.9411 -1.9334

-
1
.
7
9
4
1

-
1
.
7
8
6
0

-
1
.
7
7
5
5

.
8
6
9
2

-
1
.
8
4
2
5

7
-
1
.
9
4
1
3

-
1
.
9
3
9
8

-
1
.
9
3
9
0

-
1
.
9
2
8
1

-
1
.
9
2
4
5

-
1
.
9
2
2
3

-
1
.
9
2
0
9

-
1
.
9
2
0
0

-
1
.
5
1
8

-
1.5
94
2-
1.5
86
6
-
1.3
94
3
-
1.2

2.0
07
5-
1.9
87
4-
1.9
72
1-
1.9
60
9-
1.9
52
9

0

0.2
0.4
0.6
0.8
•

1

-
1
.
9
4
7
3

-
1
.
9
4
3

717 -1.5661 -1.5621

1	6	-
.7	4	1
6	-	.
8	1	5
5	.	5
	6	9
	7	3
	4	
	9	-
	-	1
	1	5
	.	5
	6	7
	7	4
	4	-
	0	1
	-	.
	1	5
	.	5
	6	6
	7	2
	3	
	5	-
		1
		.
		5
		5
		5
		-
		1
		.
		5
		4
		8
		-
		1
		.
		4
		1
		7
		1
		-
		1
		.
		4
		1
		6
		5
		-
		1
		.
		6
		7
		8
		6
		-
		1
		.
		6
		2
		7

- 1 . 6 7 3 2

-2

-	1	
.	8	
1	6	
1		
-	1	
.	7	
9.8		
0		
8		
-	1	
.	7	
6		
7		
6		
-	1	
.	6	
1	9	
.	3	
7		
4		
6		
9		
-	1	
.	6	
1	4	
.	9	
7	6	
2		
9		
2		
-	6	
1.7	3	
14	1	
5-	3	
1.7		
02	-	
8-	1	
1.6	.	
93	6	
7-	1	
1.6	4	
86	8	
9-		
1.6	-	
82	1	
0	.	
	6	
	0	
	0	
	6	
-		
1.5		
88		
7-		
1.5		
79		
1-		
1.5		

-1.5068 -1.4915 -1.4772 -1.4642 -1.4528
-1.4433 -1.4356 -1.4296 -1.4251 -1.4219

- 1 . 5 5 4 8

-0.5580 -0.5548 -0.5523 -0.5504 -0.5490

0 3
. 5
9 6
1 0
7 .
0 0
- 7
0 3
. 4
9 9
1 0
6 .
8 0

-
0
.
5
4
4
8
1
-
0
.
5
4
4
7
3
4
4
-
0
.
5
4
4
7
3
4
1
-
0
.
5
4
4
6
7
3
3
9
-
0
.
5
4
6
6

-
0
.
7
3
3
8

-
0.8

-
0
.
7
5
4
-
0
.
7
0.6
8
7

-0.4

-
0
.
7
6
2
3
-
0
.
5
8
3
7
5
6
4
-
0
.
7
7
3
5
1
2

-0.3895 -0.3841 -0.3790 -0.3743 -0.3701
-0.3665 -0.3636 -0.3613 -0.3595 -0.3583

-
0.7
46
8-
0.7
43
2-
0.7
40
4-
0.7
38
2-
0.7
36
7

.
5
7
1
6
-
0
.
5
6
6
5
-
0
.
5
6
1
9

-
0
.
7

-
0
.
3
5
7
4
-
0
.
3
5
6
4
-
0
.
3
5
6
2

-
0
.
5
4
6
5

-
0.2

-
0
.
1
9
4
9

-
0
.
10
8
9
9

-
0
.
1
8
5
0

-
0
.
1
8
0
5

-
0
.
1
7
6
6

0.2

0
.
0
0
5
0

0
.
0
0
9
7

0
.
0
1
4
1

0
.
0
1
8
1

0
.
2
0
4
9

0
.
2
0
9
4

0
.
2
1
3
4

0.2168 0.2196 0.2218 0.2235 0.2247

-
0.1
73
2-
0.1
70
4-
0.1
68
2-
0.1
66
6-
0.1
65
4

0.0
21
4
0.0
24
1
0.0
26
3
0.0
27
9
0.0
29
1

-
0
.
3
5
6
1

1
6
4
0

-
0
.
1
6
3
6

-
0
.
1
6
3
4

-
0
.
1
6
3
3

3
0
5

0
.
0
3
0
8

0
.
0
3
1
0

0
.
0
3
1
2

0
.
0
3
3

-
0
.
1
6
4
5

-
0
.
1
6
4
5

0
.
0
2
9
9

0
.
0
0

0
.
2
2
5
5

0
.
2
2
6
1

0
.
2
2
6
4

0
.
0
2
9
2
6
7

0

-
0
.
1
6
3
2

08
7
0.6
12
0
0.6
14
6
0.6
16
5
0.6
18
0

0.4
0.4
04
8
0.4
09
0

0.4
12
7
0.4
15
6
0.4
18
0
0.4
19
8
0.4
21
0

1

1.0045 1.0081 1.0108 1.0128

.
2
2
6
8

0
.8
1
7
7
0
.8
1
8
0
0
0
8
2

0
.6
1
9
0
0
.6
1
9
7

0
.6
2
2
0
1
0
0
6
2
0
1
4
1

0
.4
2
2
1
9
0
.4
2
2
2
6
0
.4
2
2
9
0
.4
2
3
2
0
.4
2
3
3

0
1
4
1
1
0
5
1
1
0
1
5
7
1
0
1
6
0
1
0
1
6
3

0
.8
1
6
4
0
.8
1
7
2

0
.2
2
6
9

0
.4
2
3
4

0.6
0.6
04
7
0.6

1
.2
0
4
4
1
.2

1.8

0
7
8
1
2
1
0
3

0
9
8
1
4
1
1
2

1
.8
0
4
2
1
.8
0
7
0
1
.8
0
8
9
1
.8
1
0
0
1
.8
1
0
7

1
.2
1
2
0
1
.2
1
3
1
1
.2
1
3
9
1
.2
1
4
3

1
.4
1
2
2
1
.4
1
2
8
1
.4
1
3
2

1.6

2

1
.2
1
4
6

1
.6
0
7
3

2
.0
0
4
1
2
.0
0
6
8
2
.0
0
8
5
2
.0
0
9
4

1
.2
1
4
8

1.4

V
a
l
u
e
s

n
o
t

1
.6
0
9
3

1.4
04
3
1.4
07
5

2.2

1.6
04
3

1
.4

2
.2
0
4

2.4
2.4
04
0
2.4
06
3
2.6
2.6039
2.8

0
2
.
2
0
6
6
2
.
2
0
8
1

2
.
2
0
8
9
2
.

0 2.5
67
4
-
2.5
38
5-
2.4
99
1-
2.4
48
7-
2.3
88
2-
2.3
19
6

-
2.3
87
4

-
2.5
87
2-

-		3
2		7
.		-
3		1
5		.
6		9
9		5
		9
-		7
2		
.		-
2		1
6		.
1		8
1		9
		8
-		4
2		
.		-
1	-	2
7	2	.
0	.	0
2	2	3
	4	0
-	5	7
2	7	
.		-
0	-	1
8	2	.
5	.	9
5	1	8
	6	0
-	9	9
2	5	
.		-
0	-	1
0	2	.
8	.	9
0	0	2
	9	8
-	3	1
2	8	
.		-
3	-	1
1	2	.
3	.	8
0	0	7
	2	4
-	0	4
2	7	
.		-
2	-	1
2	1	.
5	.	8
6	9	2
	5	1
-	2	8
2	1	
.		-
1		1
4		.
0		8
9		8
		7
-	2	5
2	.	
.	1	-
0	5	1
6	1	.
0	4	8
9		4
	-	9
-	2	3
.	.	
9	0	-
8	8	1
6	8	.
9	4	8
		0
	-	7
	2	7
	.	
	0	-
	2	1

1.2
1.4
1.6
1.8
•
2.2
2.4
2.6
2.8
3
-3

2

-
1
.
9
3
8
5
-
1
.
8
7
7
2
-
1
.
2.8
2
4
3
-
1
.
7
7
9
6
-
1
.
7
4
2
7

V
a
l
u
e
s
n
o
t
a
l
l
o
w
e
d

-1.9200 -1.8607 -1.8093 -1.7658 -1.7298
-1.7008 -1.6780 -1.6607 -1.6479 -1.6388

-
1.5
88
4 -
1.5
73
4
-
1.3
88
6
-
1.2
•
0.2
0.4
0.6
0.8
•

0

1

.
7
6
4
3
-
1
.
7
2
0
7

2

-
1
.
6
3
3
6
5
-
1
.
6
3
4
8
-
1
.
6
3
3
8

-
1
.
6
3
2
5
-
1
.
6
2
8
3

-
1
.
6
2
5
6
-
1
.
6
2
3
9

-
1
.
6
4
3
5

-
1
.
6
3
9

-
1
.
1
0
6
2

044 -1.3958 -1.3895

1	9	-
.5	0	1
2	-	.3
9	1	8
4	.4	5
	6	2
	6	-
	9	1
	-	.3
	1	8
	.4	2
	6	3
	5	-
	6	1
	-	.3
	1	8
	.4	0
	6	5
	4	-
	8	1
		.3
		7
		9
		3
		-
		1
		.3
		7
		8
		6
		-
		-
		1
		.2
		7
		6
		3
		-
		1
		.2
		7
		3
		8
		-
		1
		.2
		7
		2
		2
		-
		1
		.2
		7
		1
		2
		-
		1
		.2
		7
		0
		6
		-
		1
		.4
		6

- 1 . 4 6 4 4

-2

- 1
. 6
7 8
6
- 1
. 6
3.8
9 3
- 1
. 6
0 3
8
- 1
. 5
2
5 1
7 2
7 1
- 1
. 5
2 9
. 3
5 4
6 3
- 1
. 4
9 9
2 2
1.5
24 6
- 1.5
07 4
- 1.4
94 1
- 1.4
84 2
- 1.4
77 2
- 1.4
31 1
- 1.4
16 0
- 1.4

-1.4251 -1.3982 -1.3733 -1.3510 -1.3318
-1.3158 -1.3029 -1.2929 -1.2854 -1.2801

- 1 . 3 7 8 2

6-1.0047

-
1.4

-
1
.
2
7
1
8

-
1
.
2
4
9
8

-
1
.
2
9
2

-
1
.
2
1
0
7

-
1
.
1
9
4
6

-
1.1
81
1-
1.1
70
2-

-
1.1
61
7-
1.1
55
3-
1.1
50
7

-
1
.
4
1
9

-
1.0
30
5-
1.0
21
3-
1.0
14
1-
1.0
08

-
1
.
0
6
2
-
1
.
8
8
1
-1
1
.
7
1
0
-
0
4
1
9

-0.9318 -0.9166 -0.9021 -0.8890 -0.8774
-0.8677 -0.8597 -0.8535 -0.8488 -0.8455

.
1
4
4
0
-
1
.
1
4
3
2
-
1
.
1
4
2
7

-
1
.
0
0
2
0
-
1
.
1
0
.
9
9
9
0
-
0
.
9
9
8
2
-
0
.
9
9
7
8

-
1
.
1
4
7
5
-
1
.
1
4
5
4
-
1

-
0
.
8
4
3
1
-
0
.
8
4
1
5
-
0
.
8
4
0
5
-

-
1
.
2
7
0
3

-0.5169 -0.5107 -0.5058 -0.5021 -0.4994

0
.8
3
9
9
-
0
.8
3
9
5

7
4
1
-
0
.6
7
2
7
-
0
.6
7
1
8
-
0
.6
7
1
2
-
0
.6
7
0
9
-
0
.4
9
4
4
7

-
0
.6
7
0
7

-
0.8

-
0
.7
5
1
2
-
0
.7
8
6
1

-0.4

-
0
.7
2
5
6
-
0
.6
7
1
4
2
-
0
.7
0
4
1

-0.3790 -0.3683 -0.3581 -0.3488 -0.3405
-0.3334 -0.3276 -0.3231 -0.3197 -0.3172

-
0.6
95
6-
0.6
88
7-
0.6
83
2-
0.6
79
1-
0.6
76
2

.5
4
3
7
-
0
.5
3
3
5
-
0
.5
2
4
6

-
0
.6

-
0
.3
1
5
4
-
0
.3
1
4
3
-
0
.3
1
3
5
-
0
.3
1
3
1

-
0
.4
9
4
5

-
0.2

-
0
.1
8
9
9
-
0
.10
7
9
8

0.2

0
.0
0
9
9
0
0
1
9
5
0
0
2
8
3
0
0
3
6
1

0
.2
0
9
7
0
.2
1
8
8
0
.2
2
6
8

0.2336 0.2392 0.2437 0.2470 0.2495

-
0.1
46
5-
0.1
41
0-
0.1
36
6-
0.1
33
3-
0.1
31
0
0.0
42
8
0.0
48
3
0.0
52
6
0.0
55
8
0.0
58
2

-
0
.3
1
2
8
-
0
.1
2
7
4
-
0
.1
2
7
0
-
0
.1
2
6
7
6
1
0
0
6
1
7
0
0
6
2
2
1
0
0
6
2
4

0
.2
5
1
2
0
.2
5
2
3
0
.2
5
3
1
0
0
5
9
9
0
0
0
0
0
0

-
0
.1
2
6
6

17
4
0.6
24
1
0.6
29
3
0.6
33
3
0.6
36
2

0.4
0.4
09
6
0.4
18
1

0.4
25
4
0.4
31
4
0.4
36
1
0.4
39
7
0.4
42
3

1

1.0091 1.0162 1.0217 1.0257

.
2
5
3
8

0
.8
3
5
7

0
.6
3
8
2

0
.8
3
6
3

0
.6
3
9
5

0
.6
4
0
4

1
.0
2
8
5

0
.4
4
4
1

0
.6
4
0

1
.0
3
0
4

0
.4
4
5
3

0
.6
4
1
3

1
.0
3
1
6

0
.4
4
6
1

0
.6
4
6

1
.0
3
2
3

0.8
0.8
09
2
0.8
16
8
0.8
22
8
0.8
27
4
0.8
30
7

1.2

0.6
0.6
09
4
0.6

1
.2
0
8
9
1
.2

0
.8
3
3
1

0
.8
3
4
6

0
.2
5
3
9

0
.4
4
7
1

6
1.8

1
5
7
1
.2
2
0
6

4
1
9
6
1
.4
2
2
7

1
.8
0
8
4
1
.8
1
4
1
1
.8
1
7
8
1
.8
2
0
2
1
.8
2
1
5

1
.2
2
4
4
1
1
.2
2
6
5
1
.2
2
8
0
1
.2
2
8
9
1
.2
2
9
5

1
.6
1
4
6
1
.6
1
8
7
1
.6
2
1
4
1
.6
2
3
0
1
.6
2
4
0

2
.0
0
8
3
2
.0
1
3
7
2
.0
1
7
0
2
.0
1
9
1

1
.2
2
9
8

1.6

2

V
a
l
u
e
s

n
o
t

a
l
l
o
w
e
d

1.4

1.4
08
7
1.4
15
1

2.2

1.6
08

1
.

2
.2
0

2.6
2.6078

2.4
2.4
08
0
2.4
12
8

2.8

8
1

2
.
2
1
3
2

2
.
2
1
6
3

0 2.2
13
2
-
2.0
17
0-
1.8
20
2-
1.6
23
0-
1.4
25
9-
1.2
28
9

-
2.2
08
1

-
2.4
08
0-

-		0
1		4
.		-
0		0
3		.
3		4
0		4
		6
-		1
0		-
.		0
8		.
3		2
6		5
9		3
		1
-		-
0		1
.		.
6		0
4	1	3
1	.	0
5	0	3
	3	0
-	2	4
0	3	-
.		0
4	-	.
4	0	8
7	.	3
1	8	4
	3	7
-	6	
0	3	-
.		0
2	-	.
5	0	6
3	.	3
9	6	9
	4	5
-	1	5
1	0	-
.		0
0	-	.
3	0	4
2	.	4
8	4	5
	4	3
-	6	-
0	6	0
.		.
8	-	2
3	0	.
6	.	2
7	2	5
	5	2
-	3	3
0	5	-
.		1
6		1
4		.
1		0
3		2
		8
-	1	5
0	.	-
.	0	.
4	3	0
4	1	.
6	6	8
9		3
		3
-	0	1
0	.	-
.	8	.
2	3	0
5	5	.
3	7	6
8		3
		8
	0	2
	.	-
	6	.
	4	0

1 1.1454 1.2738 1.3823

3
1
5
7
5
6
1
6
0
5
5
1
6
2
3
9
.
4
6
6
9
1
.
5
3
1
7
1
.
5
7
7
2
1
.
6
0
7
1
1
.
6
2
5
6
1
.
4
6
4
6
9
0
1
.
5
3
5
3
4
0
1
.
5
7
5
7
9
7
1
.
6
0
9
7
1
.
6
2
8
3

1
.
6
3
4
8

-
2.4

-
0
.
0
6
1
7
0
2.2
1
2
7
4
0
.
3
1
3
5
0
.
4
9
5
5
0
.
6
7
1
8
0.8
40
5
0.9
99
0
1.1
44
0
1.2
72
2
1.3
80
5
00

-2

-0.0599 0.1293 0.3154 0.4975 0.6741
0.8431 1.0020 1.1475 1.2763 1.3852

1
.
4
7
2
3
1
.
5
3
7
6
1
.
5
8
3
5
1
.
6
1
3
8
1
.
6
3
2
5

1

-
1.8

-
0
.0
5
8
2
0
1.6
1
3
1
0
0
3
1
7
2

-
0
0
5
5
8
0
1.4
1
3
3
9
9
4
0
1
9
7
6
0
2

-0.0526 0.1366 0.3231 0.5058 0.6833
0.8535 1.0141 1.1617 1.2929 1.4044

0.8
45
5
1.0
04
7
1.1
50
7
1.2
80

0.8
48
8
1.0
08
6
1.1
55
3
1.2
85
4
1.3
95
8

-1.2

-0.0483 0.1410 0.3276 0.5107 0.6887
0.8597 1.0213 1.1702 1.3029 1.4161

1
.4
7
2
1
.5
4
3
0
1
.5
8
9
3
1
.6
1
9
9

1
.6
3
8
8
6
2
8
7
1
.6
4
4
7
9

1
.4
9
4
1
1
5
6
1
8
1
.6
0
9
5
1
6
4
1
1
6
0
7

1
.4
8
4
2
1
1
5
0
7
4
1
1
.

4
3
4

1

1.0419 1.1946 1.3318 1.4499

5		
7	1	1
6	.	.
5	5	5
	2	4
1	4	6
.	6	3
6		
2	1	1
5	.	.
4	5	6
	9	2
1	5	0
.	7	0
6		
5	1	1
7	.	.
8	6	6
	4	7
1	6	2
.	3	7
6		
7	1	1
8	.	.
0	6	7
	7	0
	9	7
	8	8
	1	1
	.	.
	7	7
	0	2
	0	9
	8	8

1
.6
8
9
8

-1

-	
0	
.	
0	
4	
2	
8	
0	-0.6
0.8	
1	
4	
6	
5	
0	
.	
3	
3	
3	
4	
0	
.	
5	
1	
6	
9	
0	
.	
6	
9	
5	
6	
0.8	
67	
7	
1.0	
30	
5	
1.1	
81	
1	
1.3	
15	
8	0.8
1.4	77
31	4

-0.0283 0.1611 0.3488 0.5335 0.7142
0.8890 1.0555 1.2107 1.3510 1.4727

1
.5
7
2
7

1
.6
4
9
8

1
.7
0
5
2

1
.7
4
2
4

1
.7
6
5
8

0.9
47
2
1.1
24
7
1.2
94
4
1.4
53
1
1.5
96
7

8
1.4
81
1
1.6
31
8

8
2
0
7
6
4
5

0.8

0.9891 1.1754 1.3575 1.5329 1.6983

0
2
3
7
2
0
9
3
8
2
1
4
0
9
1
9
5
9
7

1
7
2
0
7
1
8
2
1
8
1
9
8
8
4
1
9
5
2
1
1
9
8
6
9

1
7
6
4
3
1
8
4
4
1
9
5
9
7
2
0
2
0
7
2
0
6
0
9
2
0
8
8
4
2
1
6
9
5
2
8
2
5
6

0.6
0.7
89
3
0.9
76
4
1.1
60
0
1.3
38
1
1.5
08
0
1.6
66
1

1

1.1888 1.3744 1.5548 1.7272

0.4
0.5
89
5
0.7
77
3
0.9
62
3
1.1
42
9
1.3
16

1
8
0
7
7
1
9
2
8
1
2
.

1
8
7
5
2
0
3
0

2
0
0
8
0

1.4

V
a
l
u
e
s

n
o
t

1.2 1.5
88
4
1.7
72
3

1.7882
1.8

1
.3
8
8
6

1
.5
7
3
4

1
.7
5
2
0

7 0 2
2 7 2
.1
5 4
1 5
4 7

2 2
.2
4 3
5 0
7 7
0 0

2 2
.3
1 4
3 8
0 7

1 1
.9 1
4 9
2 0 8
0 7
2 9

1 1 2
2 1
9 4 1
2 3 7
2 2 0
2 2 0

1 2 2
.9 6 3
2 2 4
1 7 2
0 2 2

2 3 2
.0 8 4
7 8 9
5 2 9
7 2 1

2 4 2
.2 8
1 5
0 7
3 6

2 1
.3 9
1 7
9 1
6 1

2 2 2
.4 9 1
0 7 1

2
.3
5
6
9

1.6

2

2.2

2.4
2.5872 2.7660

2.6
2.7869

2.8

8
7
7

2

.

3

6

8

7

2

.

5

3

8

5

2

.

6

9

0

7

2

.

3

8

7

4

2

.

5

6

7

4

6

2

.

0 2.2
06
6
-
2.0
08
5-
1.8
10
0-
1.6
11
4-
1.4
12
8-
1.2
14
3

-
2.2
04
0

-
2.4
04
0-

-		0
1		1
.		-
0		0
1		.
6		4
4		2
		2
-		9
0		-
.		0
8		0
1		.
8		2
3		2
		6
-		4
0		-
.	6	1
2	1	.
0	0	0
6	1	1
	6	5
-	0	1
0	0	-
.	-	0
4	0	.
2	0	8
3	.	.
4	8	1
	1	7
-	8	2
0	0	-
.	-	0
2	0	.
2	0	6
6	6	1
9	2	9
	0	6
-	4	-
1	4	0
.	-	.
0	0	4
1	.	2
6	4	2
3	2	2
	3	6
-	2	-
0	2	0
.	-	.
8	0	2
1	.	.
8	2	2
2	2	6
	2	1
-	6	-
0	7	1
.		-
6		1
2		.
0		0
5		1
	-	4
-	1	1
0	.	-
.	0	0
4	1	.
2	5	8
3	7	1
3		6
	-	4
-	0	-
.	8	0
2	1	.
2	7	6
6	7	1
8		9
	-	0
	0	-
	.	-
	6	-
	2	0

6 1.2625 1.4181 1.5574

1	.	1
1	6	7
.	7	.
8	4	6
4	9	7
0	1	6
1	.	4
.	7	1
1	7	.
.	0	7
8	1	7
8	.	1
9	1	9
3	.	.
.	8	1
1	4	.
.	1	8
9	3	4
2	.	3
0	1	3
9	.	.
.	8	1
.	9	8
.	0	.
.	6	9
.	.	2
.	1	7
.	.	.
.	9	1
.	2	.
.	2	9
.	3	2
.	.	4
.	.	4
.	.	5

1
.9
3
9
8

-
2.4

-
0
.
0
3
0
8

0
2.2

-2

-	0	-0.0299	0.1645	0.3574	0.5481	0.7356			
3	0						0.9187	1.0956	1.2637
6	0								1.4196
0	3								1.5593
.	0								
3	5								
5	0								
6	0								
4	.								
0	1								
.	6								
5	4								
4	0								
7	0								
0	.								
0	3								
.	5								
7	6								
3	8								
4	0								
4	.								
.	5								
0.9	4								
17	7								
3	4								
1.0	.								
93	0								
9	.								
1.2	7								
61	3								
7	4								
1.4	9								
17	.								
1	0.9								
1.5	17								
56	9								
2	1.0								
	94								

0.9
17
3
1.0
93
9
1.2
61
7
1.4
17
1
1.5
56
2

1

1
9
4
1
3

-
1.8

-
0
.0
2
9
1
0
1.6
1
6
5
4
0
3
5
8
3

-
0
.0
2
7
9
0
1.4
1
6
6
0
3
5
9
5
0
5
7
3
6
7

-0.0263 0.1682 0.3613 0.5523 0.7404
0.9243 1.1023 1.2718 1.4296 1.5717

0.9
20
0
1.0
97
1
1.2
65
5
1.4
21
9
1.5
62
1

0.9
21
8
1.0
99
3
1.2
68
2
1.4
25
1
1.5
66
1

-1.2

-0.0241 0.1704 0.3636 0.5548 0.7432
0.9276 1.1062 1.2767 1.4356 1.5791

1
.6
8
2
0
1
.7
7
8
5
1
.8
5
0
9
1
.9
0
1
1

1
.9
3
3
4
8
5
7
5
1
.9
0
8
3
1
.9
4
4
1
1
1

1
.6
9
3
7
1
.7
9
2
4
1
.8
6
6
7
1
.9
1
1
8
5
1
9

1
.6
8
6
9
1
7
0
2
8
1
1
1
1

4
7
3

7

1.1175 1.2905 1.4528 1.6006

8	1	1
0	.	.
3	7	7
2	1	2
	4	9
1	5	2
.		
8	1	1
7	.	.
9	8	8
1	1	3
	7	4
1	3	9
.		
9	1	1
3	.	.
2	8	9
2	9	1
	5	5
1	3	8
.		
9	1	1
6	.	.
6	9	9
6	5	7
	0	3
	2	0
	1	2
	.	.
	9	0
	8	1
	5	0
	8	4

1
.9
8
7
4

-1

-	
0	
.	
0	
2	
1	
4	
0	-0.6
0.8	
1	
7	
3	
2	
0	
.	
3	
6	
6	
5	
0	
.	
5	
5	
8	
0	
0	
.	
7	
4	
6	
8	
0.9	
31	
8	
1.1	
11	
3	
1.2	
82	
9	
1.4	
43	
3	0.9
1.5	37
88	0

-0.0141 0.1805 0.3743 0.5665 0.7564
0.9430 1.1248 1.2995 1.4642 1.6148

1
.7
4
6
9

1
.8
5
6
4

1
.9
4
0
9

2
.0
0
1
1

2
.0
4
0
8

2
.0
0
7
5

0.9
73
3
1.1
61
5
1.3
45
3
1.5
22
4
1.6
89
6

2
1.5
37
8
1.7
09
8

9
0
0
7
8
2
1

0.8

0.9945 1.1876 1.3784 1.5656 1.7469

1
7
4
1
2
2
7
6
7
2
3
5
1
6
9
2
1
2
9
5
2
1
9
2
2
0
7
8
0
2
2
1
7
7
2
1
6
8
4
2
2
5
9
1
8
9
5
1
2
4
5
3
2
.

0.6
0.7
94
6

0.9
88
1
1.1
79
8
1.3
68
4
1.5
52
4
1.7
29
2

1

1.1944 1.3871 1.5770 1.7624

0.4
0.5
94
7
0.7
88
6
0.9
81
0
1.1
71
0
1.3
57

2
2
6
2
5

1.4

V
a
l
u
e
s

n
o
t

1.2 1.5
94
2
1.7
86
0

1.7941
1.8

1
.3
9
4
3

1
.5
8
6
6

1
.7
7
5
5

8
2
.2
5
8
4

2
.3
8
6
2

2
.4
8
6
1

1
2

4
2
.1
7
2
0

2
.3
5
1
0

2
.5
1
7
2

2
.6
6
4
0

1
9
7
3
8

2
1
5
5
2

2
.3
2
5
9
9
0

2
.4
8
0
3

2
.6
1
1
3

2
.4
3
6
2

1
9
8
5

2
.7
1
2
9

2
.8
5
6
8

1.6

2

2
.1

2.2

2.4
2.5935 2.7826

2.6
2.7934

2.8

9
3
8
2
.
3
8
4
1
2
.
5
6
8
1
2
.
7
4
1
3

2
.
3
9
3
7
2
.
5
8
3
3
2
.
7
6
5
8

2
.

8
9
6
8

0 2.2
02
6
-
2.0
03
4 -
1.8
04
0 -
1.6
04
5 -
1.4
05
1 -
1.2
05
7

-
2.2
01
6

-
2.4
01
6 -

-		8
1		0
.		-
0		0
6		4
5		0
		9
-		2
0		-
.		0
8		0
0		2
7		1
3		0
		6
-		-
0		1
.	-	1
6	1	0
0	.	0
8	0	0
2	0	6
	6	0
-	4	0
0		-
.	0	0
4	0	8
0	.	0
9	8	6
3	0	6
	7	9
-	2	-
0		0
.	-	-
2	0	0
1	.	6
0	6	0
7	0	7
	8	8
-	1	-
.	-	0
0	0	.
0	.	4
6	4	0
5	0	9
	9	0
-	2	-
0		0
.	-	.
8	0	2
0	.	1
7	2	0
2	2	4
	1	-
-	0	1
0	6	-
.		1
6		0
0		0
8		0
2		5
	-	6
-	1	-
0	.	-
.	0	0
4	0	.
0	6	8
9	2	0
3		6
	-	5
-	0	-
.	8	0
2	0	.
1	7	6
0	0	0
7		7
	-	6
	0	-
	.	-
	6	0
	0	0

3 1.3419 1.5211 1.6909

4 . 1
2 8 4
. 4 6 8
0 1 4
9 6 6
5 1 9
4 . 1
2 9 1
. 8 .
1 3 9
8 0 8
1 2 4
6 0 0
2 9 2
. 6 0
2 2 9
4 2 7
1 2 5
9 . 2
8 1 1
2 2 8
6 6 4
2 2 4
. 2 2
2 4 2
4 3 2
0 0 4
4 4 4
8 8 8

2
. 2
8
0
5

-
2.4

-
0
. 0
1
2
3

0
2.2

1 -2

8 0 -0.0120 0.1858 0.3829 0.5790 0.7739

5 0 0.9667 1.1567 1.3425 1.5218 1.6918

4 . 0

0 1

. 2

3 2

8 0

2 5

5 . 1

0 8

. 5

5 6

7 0

8 6

6 . 3

0 8

. 2

7 6

3 0

4 . 5

0.9

66

1

1.1

56

0

1.3

41

6

1.5

20

7

1.6

90

3

0.9

66

4

1.1

56

1

-
1.8

-
0
0
1
1
6
0
1.6
1
8
6
1
0
3
8
3
2
0
5
7
9
4
0
7
7
4
3

-
0
0
1
1
2
0.4
1
8
6
0
3
8
3
7
0
5
8
0
0
0
7
7
4
9

-0.0105 0.1873 0.3844 0.5808 0.7758
0.9690 1.1596 1.3461 1.5265 1.6979

0.9
67
3
1.1
57
4
1.3
43
3
1.5
22
9
1.6
93
2

0.9
68
0
1.1
58
3
1.3
44
5
1.5
24
4
1.6
95
2

-1.2

-0.0097 0.1881 0.3854 0.5818 0.7770
0.9704 1.1613 1.3482 1.5293 1.7015

1
8
4
9
9
1
9
8
8
0
2
1
0
2
5
2
1
9
0
2

2
2
5
1
7

1
0
6
8
2
1
9
5
4
2
2
5
7
7

1
8
5
6
2
1
9
9
6
2
2
1
1
2
8
2
2
7

1
8
5
2
5
1
9
1
4
2
2

1
8
6
1
0
2

8
6
6

2

1.1660 1.3542 1.5371 1.7120

0	1	1
0	.	.
2	8	8
5	6	7
2	7	4
.	2	9
1	2	2
2	.	.
0	0	0
9	1	2
2	0	0
.	7	9
2	2	2
1	.	.
2	1	1
6	3	4
.	1	4
2	4	5
.	.	.
2	2	2
7	.	.
7	2	2
5	2	2
.	2	4
.	5	1
.	4	5
.	.	.
.	2	2
.	.	.
.	2	3
.	9	1
.	2	1
.	3	2

2
.3
1
9
4

-1

-
0
.
0
0
8
6
0
0.8
1
8
9
3
0
.
3
8
6
5
0
.
5
8
3
1
0
.
7
8
5

-0.6

-0.0057 0.1922 0.3897 0.5865 0.7824
0.9768 1.1691 1.3582 1.5422 1.7188

1
.
8
8
4
0
2
.
0
3
3
1
2
.
1
6
0
3
2
.
2
6
1
2
2
.
2
3
3
4
4

0.9
72
2
1.1
63
4
1.3
50
9
1.5
32
8
1.7
06

0.9
74
3

5
8
4
7
0
.
7
8
3

-
0.4

-
0
.0
0
3
9
0
0.2
1
9
4
0
0
3
9
1
6
0
.5
8
8
6
0
.7
8
4
8
0.9
79
7
1.1
72
6
1.3
62
6
1.5
48
0
1.7
26
5
0.9
82
8
1.1
76
4
1.3
67
5
1.5
54
4
1.7
35
1

-
0
.0
0
2
0
0
1
9
5
9
0
3
9
3
6
0
5
9
0
9
0
7
8
7
4

0.1980 0.3958 0.5932 0.7901
0.9860 1.1804 1.3725 1.5611 1.7441

0.2

0
.3
9
7
9
0
.5
9

1
.8
9
4
5
2
.0
4
7
2
2
.1
7
8
9
2
.2
8
4
6

2
.3
6
2
2
1
9
9
9
2
.3
1
1
3
2
.3
9
4
5
1
.9
1
8
5
2
.0
8
0
0
2
2
2
2
8
2
.3
4
1
1
2
.4
3
1
0

8
2
7

0.9
89
2
1.1
84
4
1.3
77
6
1.5
67
8
1.7
53
3

6
1.5
74
4
1.7
62
2

5
6
0
7
9
2
8

0.8

0.9978 1.1950 1.3913 1.5860 1.7781

1
9
3
1
2
2
0
9
7
6
2
2
4
6
9
2
3
7
3
0
2
4
7
1
0

2
9
5
3
2
4
3
9
3
2
5
5
7
5

0.6
0.7
97
8

0.9
95
2
1.1
91
8
1.3
87
2
1.5
80
5
1.7
70
6

1

1.1978 1.3948 1.5907 1.7847

1
9
5
5
5
2
1
3
1
9
2
.

0.4
0.5
97
9
0.7
95
4
0.9
92
3
1.1
88
3
1.3
82

2
5
3
9
9

1.4
V
a
l
u
e
s

n
o
t

1.2 1.5
97
7
1.7
94
4

1.7976
1.8

1
.3
9
7
7

1
.5
9
4
6

1
.7
9
0
1

9 0 1
2 1 2
.3 1
3 8
7 8
7 6

2 2
.5 3
0 7
0 9
5 7

2 2
.6 5
4 6
2 4
4 7

1 1
.9 8
8 9
9 4
4 7
2 6
2 1
1 8
1 1
5 9
2 3
8 8

1 3 2
.9 6 3
8 8 8
3 8 7
2 2 8

2 5 2
.4 4
1 7 5
7 7 7
2 2 6
3 2

2 7 2
.3 1
5 2 7
4 7 6
8 0 1

2 1
.5 9
2 6 2
2 2
2 1
2 9 2
.6 9 1
8 4

2
.7
5
6
3

1.6
2

2.2

2.4
2.5974 2.7929

2.6
2.7973

2.8

9
7
5

2

.

3

9

3

6

2

.

5

8

6

9

2

.

7

7

5

4

2

.

3

9

7

5

2

.

5

9

3

3

2

.

7

8

5

9

2

.

9
5
4
9

Lower
Truncation
Point,
kL

1

2.0

0
.
2

5.0

10.
0

2 0
0
.
5

1
.
0

5
0
.
0

C
o
m
b
i
n
a
t
i
o
n
s

20.0

100
.0

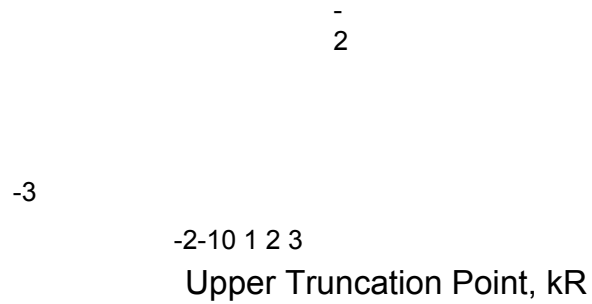


Figure 3 – Contour Plot of the Magnitude of the Coefficient of Variation of the Doubly-Truncated Normal Distribution (in Terms of the Standard Normal Variate)

Use of the Tables:

An individual is dealing with a pattern of variability (random variable) that is normally distributed with a mean value of 20 units and a standard deviation of 5 units. Due to additional constraints (e.g., previous sorting/inspection), the actual values of this distribution are restricted to the range from 12 units to 32 units. For planning purposes, he would like to estimate the 95th percentile of this demand pattern.

This demand pattern would be appropriately modeled as that of a doubly-truncated normal distribution. Using the notation of Figure 1: $\mu = 20$, $\sigma = 5$, $x_L = 12$, and $x_R = 32$. In terms of the standard normal distribution, the truncation points k_L and k_R can be calculated using Equation 5

$$k_L = \frac{\bar{x}_L - \mu}{\sigma} = -1.6 \text{ and } k_R = \frac{\bar{x}_R - \mu}{\sigma} = 2.4$$

Using Table 4, with $k_L = -1.6$ and $k_R = 2.4$, one finds that the standard z value associated with $F_{DTN}(z) = 0.95$ is $z = 1.5978$. This value, in turn, corresponds to a value of $x = 20 + 1.5978(5) = 27.989$ units with respect to the original distribution. This can be compared to the value of 28.225 units that would be obtained by using (non-truncated) standard normal tables.

Conclusion

This paper has presented tables of the cumulative distribution function of the doubly-truncated normal distribution for use by workers/researchers in management science, finance, and the many other fields in which this family of distributions has found application. In the process of developing these tables, the characteristic parameters of these distributions (in terms of the standard normal variate z) were used to calculate the corresponding coefficients of determination. It was noted that, while the upper and lower truncation points solely determine the coefficient of variation, this determination is not a one-to-one distribution – unlike the case for singly-truncated normal distributions.

References

- ABRAMOWITZ, M. and I. STEGUN (1972), Handbook of Mathematical Functions, Dover Publications, New York. BARR, D. and E. SHERRILL (1999), "Mean and Variance of Truncated Normal Distributions." The American Statistician, 53, 357-361. COHEN, A. (1991), Truncated and Censored Samples, Marcel Dekker, New York. HALD, A. (1952), Statistical Theory with Engineering Applications, John Wiley, New York.

- JOHNSON, A. (2001), "On the Truncated Normal Distribution: Characteristics of Singly- and Doubly-truncated Populations of Application in Management Science," Doctoral Dissertation, Stuart Graduate School of Business, Illinois Institute of Technology, Chicago, Illinois.
- JOHNSON, N. and S. KOTZ (1970), Distributions in Statistics: Continuous Univariate Distributions 1, Houghton Mifflin, New York.
- KECEIOGLU, D. (1991), Reliability Engineering Handbook (Vol. 1), Prentice Hall, Englewood Cliffs, New Jersey.
- SCHNEIDER, H. (1986), Truncated and Censored Samples From Normal Populations, Marcel Dekker, New York.
- THOMOPOULOS, N. (1980), Applied Forecasting Methods, Prentice-Hall, Englewood Cliffs, New Jersey.