CHARACTERISTICS AND TABLES OF THE LEFT-TRUNCATED NORMAL DISTRIBUTION

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ABSTRACT

Truncated normal distributions have found utility in a wide variety of fields – including those within the purview of management science. Despite this fact, work to date has focused primarily on estimation of the original (non-truncated) population parameters based upon truncated or censored samples. This paper presents selected reference tables of the cumulative distribution function of left-truncated normal distribution for use and reference by workers/researchers. In the process of developing these tables, the characteristic parameters of this distribution are derived in terms of "standardized" truncated normal distributions. In addition, it is noted that the point of truncation uniquely determines the value of the coefficient of variation – which can, in turn, be used to estimate the truncation point based upon a sample.

INTRODUCTION

The normal – or Gaussian – distribution is one of the most widely utilized of all random variables. Mound-shaped or approximately mound-shaped distributions are encountered in a large number of applications and, *via* the Central Limit Theorem, provide the underpinning for the characteristics of sampling distributions upon which statistical inference is based. Despite this utility, the fact that the values of a normally distributed random variable can, in theory, assume any value over the range from $-\infty$ to $+\infty$ may lead to significant computational errors in situations in which the distribution's outcomes are constrained. This problem has motivated the study of truncated normal distributions.

The study of the properties of normally distributed random variables when certain outcomes are constrained or restricted has been a rich and fertile one – with applications reported in the areas of inventory management, regression analysis, and financial modeling to name only a few. Indeed, the characteristic parameters (i.e., mean μ and standard deviation σ) of truncated normal distributions can be readily derived using basic statistical methods. The vast majority of the work to date has focused on estimating the parameters of the original, non-truncated population based upon the sample statistics of the truncated or censored samples. Little work has been done in terms of describing and tabulating the properties of the truncated normal distributions themselves.

This paper presents tables of the cumulative distribution function of the left-truncated normal distribution for use by workers/researchers in management science, finance, and the many other fields in which this family of distributions has found application. In the process of developing these tables, the characteristic parameters of this distribution are derived in terms of a "standardized" truncated distribution – for maximum generality. After a brief review of prior research in this field, this paper presents a detailed development of the characteristics of the left-truncated normal distribution – used in developing the tables and visualizations presented. A brief discussion of extending this work to the right-truncated normal distribution is, then, presented.

REVIEW OF PRIOR RESEARCH

While several, different notations – often incorporating intermediately-defined functions for convenience – have been used in the literature, the notational conventions used by Hald, which describe truncated normal distributions in terms of the probability density and cumulative distribution functions of the non-truncated normal distribution, will be used in this paper.[5]

There is a large body of literature on the subject of the estimation of the parameters of the original population (μ, σ) based upon data from truncated or censored samples. Schneider provides an excellent overview of parameter estimation of truncated normal distributions in his Chapter 2.[11] One of the founders – and most prolific authors – in this field is A. Clifford Cohen at the University of Georgia, whose works span over four decades. (See, for example [4].) In general, the procedures/algorithms that have been developed depend upon whether the points of truncation are known. The focus of this paper – like that of Barr and Sherrill – is on the characteristics of the truncated population rather than the original, non-truncated population.[3]

Little work has been done in terms of generating tables – especially cumulative probability tables – and other representations for the truncated normal distribution. Hald presents a probability-paper graph of the cumulative distribution functions of three left-truncated normal distributions with degrees of truncation of 10%, 30%, and 50%.[5] Kececioglu displays the same three distributions.

[8] Johnson and Kotz provide qualitative illustrations of the probability density functions associated with the nine combinations of truncation that result from 0%, 20%, and 40% truncation on the left/lower and right/upper sides of the distribution.[7] They also provide a table of the mean and standard deviation (as well as the mean deviation divided by the standard deviation) of the truncated normal distribution for 23 selected combinations of right-and left-truncation ranging from 0 to 50%. Thomopoulos has qualitative visualizations of the left-truncated normal distribution for four truncation points and a tabular summary of the mean, standard deviation, and coefficient of variation of the left-truncated normal distribution for truncation points over a range similar to that discussed in this paper.[12] Barr and Sherrill plot the mean and variance of the left-truncated normal distribution as a function of the truncation point (degree of truncation).[3]

LEFT-TRUNCATED NORMAL DISTRIBUTION

Definition

Consider a normally-distributed random variable *x* with a probability density function *f(x)* specified as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, -\infty \le x \le \infty \quad (1)$$

If the values of *x* below some value x_L cannot be observed – due to censoring or truncation – then, as shown in Figure 1 and following Hald's conventions, the resulting distribution is a left-truncated normal distribution with probability density function $f_{LTN}(x)$ given by [5]

$$f_{LTN}(x) = \begin{cases} 0, & -\infty \le x \le x_L \\ \dfrac{f(x)}{\int_{x_L}^{\infty} f(x)dx}, & x_L \le x \le \infty \end{cases} \quad (2)$$

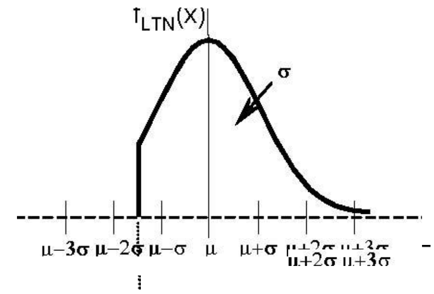
where *f(x)* is as defined in Equation 1.

For purposes of generality, Equation 2 can be re-stated in terms of the standard normal distribution (denoted *f(z)*) where

$$z = \frac{x - m}{\sigma} \quad (3)$$

and

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, -\infty \le z \le \infty \quad (4)$$



fLTN(x) x

xL

Figure 1 – Left-Truncated Normal Distribution (Original Population Parameters)

In terms of this standard normal distribution, the point of truncation x_L will be denoted k_L as given by

$$k_L = \frac{x_L - m}{\sigma} \quad (5)$$

Reformulating the left-truncated normal distribution of Equation 2 in terms of the standard normal distribution, the following can be found:

$$f_{LTN}(z) = \begin{cases} 0, & -\infty \le z \le k_L \\ \dfrac{f(z)}{\int_{k_L}^{\infty} f(z)dz}, & k_L \le z \le \infty \end{cases} \quad (6)$$

Similar expressions can be found in Cohen, Johnson and Kotz, and Schneider.[4] [7] [11] This is illustrated in Figure 2.

fLTN(z)

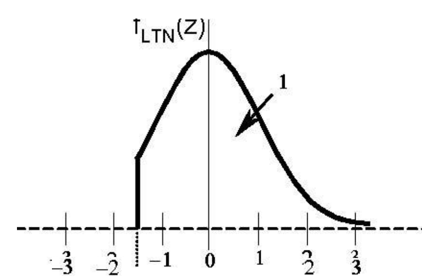
z

k_L

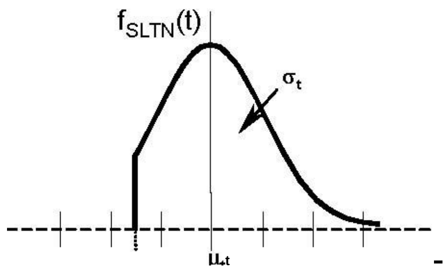
Figure 2 – Left-Truncated Normal Distribution (Standard Normal Variate)

To define what Thomopoulos terms as a "standardized, left-truncated normal distribution," a standardizing variable $t = z – k_L$ is introduced, which has the effect of defining the point of truncation as $t = 0$.[12] The standardized, left-truncated normal distribution $f_{SLTN}(t)$ is, thus, given by

$$f_{SLTN}(t) = \begin{cases} 0, & t \leq 0 \\ \dfrac{f(t+k_L)}{\int_{k_L}^{\infty} f(z)dz}, & t > 0 \end{cases} \quad (7)$$



The standardized, left-truncated normal distribution is



where *f(t)* is as defined in Equation 7.

$$\mu_t = \int_0^{\infty} t f(t)\,dt = E(t) \qquad (8)$$

Defining *H(k)* as

$$H(k) = \int_k^{\infty} f(z)\,dz \qquad (9)$$

illustrated in Figure 3.

t

0

Figure 3 – Standardized, Left-Truncated Normal Distribution

Parameters

Consider the mean (μ_t) of the standardized, left-truncated normal distribution – for a given point of truncation k_L,

and performing the integration indicated, Equation 8 can be shown to result in

1

$$\mu_t = \frac{\left[f(k_L) - k_L H(k_L) \right]}{H(k_L)} \qquad (10)$$

From Equation 10, it is clear that the mean of the standardized, left-truncated normal distribution is uniquely determined by and solely dependent upon the point of truncation.

Consider now the standard deviation (σ_t) of the standardized, left-truncated normal distribution. Given that

$$\sigma_t^2 = E(t^2) - \mu_t^2 \qquad (11)$$

It is only necessary to find $E(t^2)$.

$$E(t^2) = \int_0^{\infty} t^2 f(t)\,dt \qquad (12)$$

Performing the indicated integration, Equation 12 can be shown to result in

$$E(t^2) = \frac{\left[(1 + k_L^2) H(k_L) - k_L f(k_L) \right]}{H(k_L)} \qquad (13)$$

Equation 13 can be used along with Equation 10 to calculate σ_t as shown in Equation 11. Once again, it is worth noting that the point of truncation k_L again uniquely and solely determines the standard deviation of the standardized, left-truncated normal distribution.

Finally, it is convenient to define a coefficient of variation *c* – which, again, exists uniquely for a particular k_L

$$c = \frac{\sigma_t}{\mu_t} \qquad (14)$$

The expressions derived in this section for the mean and standard deviation of the standardized, left-truncated normal distribution can be shown to be equivalent to those presented in Schneider.[11] In particular, they represent a re-derivation of the work presented in Thomopoulos.[12] Barr and Sherrill have published an expression for the variance of the left-truncated normal distribution in terms of the Chi-Square distribution.[3]

Development of Tables

Given the formulation of the probability density function *f(t)* of the standardized, left-truncated normal distribution in Equation 7, its cumulative distribution function *F(t)* can be stated as

$$F_{SLTN}(t) = \begin{cases} 0, & t \leq 0 \\ \dfrac{F(t + k_L) - F(k_L)}{H(k_L)}, & t > 0 \end{cases} \quad (15)$$

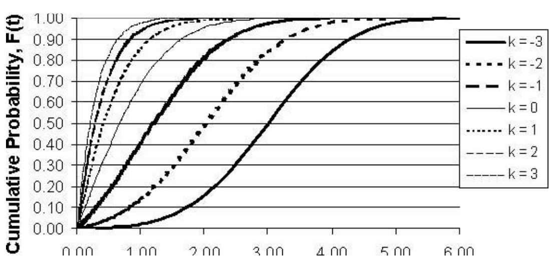
where *H(k)* is as defined in Equation 9 and *F(z)* is the cumulative probability associated with a standard normal variate of value *z*, with *H(z) = 1 – F(z)*.

Thus, given a point of truncation k_L, one can readily calculate the value of the cumulative distribution function of the standardized, left-truncated normal distribution at any standardized value of $t > 0$ through the use of Equation 15 and tables of the cumulative distribution function of the standard normal distribution *F(z)*.

With the wide availability of desktop micro-computers and their associated software, alternate methods of evaluating Equation 15 at specific values of *t* include the use of "standard" spreadsheet functions (e.g., *NormSDist(Z)*) in Microsoft Excel, user-specified functions or subroutines within Excel's Visual Basic for Applications (VBA) environment, or formulations such as those provided in Abramowitz and Stegun.[1] Based upon its ease of implementation, the Excel VBA-based approach was used in this paper.

Using the approach outlined above, a table of the cumulative distribution function *F(t)* for the standardized, left-truncated normal distribution as a function of the truncation point k_L has been developed. This table is presented in five, overlapping segments divided by point of truncation – with the range and granularity of standardized *t*'s over which *F(t)* is evaluated tailored to best illustrate the properties of the cumulative distribution function – in Johnson.[6]

Figures 4 through 6 illustrate the cumulative distribution function (CDF) of the standardized, left-truncated normal distribution function for seven values of the point of truncation (k_L = -3, -2, -1, 0, 1, 2, and 3) – over ranges of



standardize t's of 0 to 6, 0 to 3, and 0 to 1.5, respectively.

-3.0 -2.8 -2.6 -2.4 -2.2

0.5075 0.3697 0.2598 0.1784 0.1215

0.005

0.7211 0.5595 0.4196 0.3057 0.2184

0.010

0.9728 0.7954 0.6326 0.4895 0.3700

0.020

1.0621 0.8810 0.7125 0.5615 0.4323

0.025

1.1381 0.9545 0.7819 0.6250 0.4884

0.030

1.2642 1.0771 0.8990 0.7341 0.5868

0.040

1.3675 1.1782 0.9966 0.8264 0.6716

0.050

1.5692 1.3770 1.1903 1.0121 0.8460

0.075

1.7253 1.5314 1.3420 1.1594 0.9867

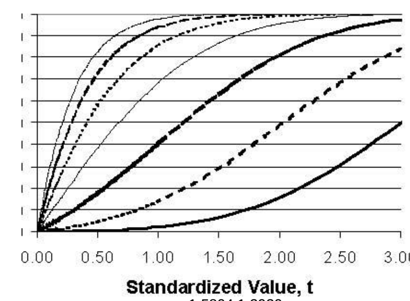
0.100

0.150
0.200
0.250
0.300
0.400
0.500
0.600
0.700
0.750
0.800
0.850
0.900
0.925
0.950
0.960
0.970
0.975
0.980
0.990
0.995

Mean Std Dev CoV

1.9685 1.7728

2.1622 1.9657 2.3287 2.1315 2.4783 2.2807 2.7487 2.5506 3.0017 2.8032
3.2547 3.0560 3.5256 3.3266 3.6755 3.4765 3.8426 3.6434 4.0373 3.8381
4.2823 4.0830 4.4402 4.2409 4.6455 4.4461 4.7513 4.5519 4.8814 4.6819
4.9605 4.7611 5.0543 4.8548 5.3269 5.1273 5.5763 5.3767
3.0044 2.8079 0.9933 0.9888 0.3306 0.3521



1.5804 1.3930

1.7716 1.5816 1.9365 1.7447 2.0850 1.8920 2.3539 2.1594 2.6058 2.4103
2.8582 2.6618 3.1284 2.9315 3.2782 3.0810 3.4450 3.2475 3.6394 3.4417
3.8842 3.6862 4.0420 3.8439 4.2471 4.0488 4.3529 4.1545 4.4829 4.2844
4.5620 4.3635 4.6557 4.4571 4.9281 4.7294 5.1774 4.9787
2.6136 2.4226 0.9820 0.9723 0.3757 0.4013

1.2130

1.3975 1.5580 1.7034 1.9682 2.2174
2.4678 2.7364 2.8855 3.0516 3.2454
3.4895 3.6469 3.8516 3.9572 4.0870
4.1659 4.2595 4.5316 4.7807

2.2360 0.9589 0.4289

the standardized value t at which the cumulative distribution function $F(t)$ assumes some value. While a table of the cumulative distribution function can, of course, be used for such purposes – subject to the granularity of the tabulated $F(t)$ values – this problem has been solved computationally through the use of a dichotomous line search algorithm, and the results are presented in Tables 1 and 2 over the range $F(t) = 0.005$ to $F(t) = 0.995$, with varying granularity. Tables 1 and 2 are presented in two segments – for negative and positive values, respectively, of the point of truncation k_L over the range -3.0(0.2)3.0. Use of these tables requires only knowledge of the truncation point k_L. An example to
illustrate the use of the tables is presented in a later section.

Cumulative Probability, F(t)

1.00
0.90
0.80
0.70
0.60
0.50
0.40

0.00 0.50 1.00 1.50 2.00 2.50 3.00

Standardized Value, t

k = -3 k = -2 k = -1 k = 0 k = 1
0.00 1.00 2.00 3.00 4.00 5.00 6.00 0.30

k = 2

Standardized Value, t

Figure 4 – CDF of the Standardized, Left-Truncated Normal

k = 30.20 0.10 0.00

Distribution for Standardized *t*'s of 0 to 6

The "inverse" of the problem of identifying the value of

Figure 5 – CDF of the Standardized, Left-Truncated Normal

the cumulative distribution *F(t)* at a standardized value *t* –

Distribution for Standardized *t*'s of 0 to 3 for a given point of truncation k_L – is that of identifying

Table 1 – Standard t Value Associated with a Cumulative Distribution Value F(t) for the Standardized, Left-Truncated Normal Distribution (k_L = -3 to 0)

-1.0 -0.8 -0.6 -0.4 -0.2

0.0

0.0172 0.0135 0.0109 0.0089 0.0074

0.0063

0.3026 0.2486 0.2061 0.1728 0.1465

0.1257

Cumulative Probability, F(z)

1.0432 0.8867 0.7461 0.6232 0.5186

1.2217 1.0570 0.9060 0.7708 0.6526 1.3783 1.2081 1.0498 0.9058 0.7774 1.5209 1.3467 1.1831

1.0322 0.8960 1.7818 1.6021 1.4310 1.2704 1.1223 2.0285 1.8450 1.6687 1.5014 1.3447

2.2770 2.0907 1.9105 1.7380 1.5746 2.5441 2.3557 2.1723 1.9954 1.8265 2.6925 2.5030 2.3183

2.1395 1.9680 2.8580 2.6676 2.4814 2.3008 2.1269 3.0512 2.8598 2.6724 2.4899 2.3135

3.2946 3.1023 2.9134 2.7290 2.5501 3.4517 3.2588 3.0692 2.8837 2.7034 3.6560 3.4625 3.2720

3.0853 2.9034 3.7613 3.5676 3.3767 3.1895 3.0068 3.8909 3.6969 3.5055 3.3177 3.1341

3.9698 3.7756 3.5840 3.3957 3.2117 4.0632 3.8688 3.6769 3.4883

3.3038 4.3350 4.1400 3.9474 3.7578 3.5719 4.5838 4.3885 4.1953

4.0048 3.8178

2.0552 1.8819 1.7174 1.5629 1.4194 0.9415 0.9197 0.8936 0.8634

0.8298 0.4581 0.4887 0.5203 0.5524 0.5846

0.4315 0.3603 0.3027 0.2564 0.2191

0.5516 0.4668 0.3966 0.3390 0.2919 0.6655 0.5696 0.4887 0.4211

0.3651 0.7752 0.6700 0.5799 0.5035 0.4392 0.9879 0.8680 0.7625

0.6708 0.5919 1.2002 1.0687 0.9508 0.8462 0.7545

1.4219 1.2810 1.1525 1.0367 0.9332 1.6669 1.5178 1.3799 1.2537

1.1393 1.8053 1.6523 1.5099 1.3787 1.2589 1.9610 1.8043 1.6575

1.5213 1.3960 2.1445 1.9839 1.8326 1.6912 1.5602

2.3778 2.2131 2.0569 1.9098 1.7724 2.5293 2.3623 2.2033 2.0531

1.9120 2.7272 2.5576 2.3955 2.2415 2.0963 2.8296 2.6588 2.4952

2.3396 2.1923 2.9559 2.7837 2.6184 2.4608 2.3112

3.0328 2.8599 2.6937 2.5349 2.3840 3.1242 2.9504 2.7831 2.6230 2.4706 3.3905 3.2144 3.0443

2.8808 2.7245 3.6350 3.4571 3.2848 3.1187 2.9593

where *f(t)* is as defined in Equation 7.

∞

μ_t $tf\ t\ dtE\ t\ \text{I}== (\ )(\ )$

0

Defining *H(k)* as

∞

$H\ k(\ )$ $=$ I $z\ dzf\ ($

1
.
2
8
7
6

67 0.6779 0.6397 0.6163 0.6471 0.6767
0.7048 0.7311

0.1891

0.2534 0.3186 0.3853 0.5244 0.6745

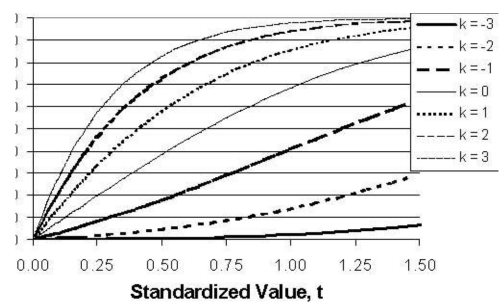
0.8416 1.0364 1.1504 1.2816 1.4395

1.6449 1.7805 1.9600 2.0537 2.1701

2.2414 2.3264 2.5758 2.8070

0.7979 0.6028 0.7555

In addition to the standardized t values associated with different values of the cumulative distribution $F(t)$ – for a given point of truncation k_L – Tables 1 and 2 also identify the mean (μ_t) and standard deviation (σ_t) associated with each truncation point. These values, which were calculated using Equations 10 and 11, respectively, were then used to calculate the coefficient of variation c associated with each truncation point. As noted earlier, this coefficient of variation is uniquely determined for the standardized, left-truncated normal distribution by the value of the truncation point k_L. As noted in Thomopoulos, the coefficient of variation c is approximately equal to 0.33 for light truncation (k_L ~ -3) and approaches 1.0 for heavy truncation (k_L ~ +3).[12]



– subject to the granularity of the coefficients of variation presented in the tables.

To assist in such estimation of the truncation point, Figure 7 plots the coefficient of variation c as a function of the point of truncation k_L.

Figure 6 – CDF of the Standardized, Left-Truncated Normal Distribution for Standardized t's of 0 to 1.5

Table 2 – Standard t Value Associated with a Cumulative Distribution Value F(t) for the Standardized, Left-Truncated Normal Distribution (k_L = 0 to 3)

2.2 2.4 2.6 2.8 3.0 0.0020 0.0018 0.0017 0.0016 0.0015 0.0039 0.0037 0.0034 0.0032 0.0031 0.0079 0.0074 0.0069 0.0065 0.0061 0.0099 0.0093 0.0087 0.0082 0.0077 0.0119 0.0111 0.0104 0.0098 0.0093

60 0.0149 0.0140 0.0132 0.0124 0.0200 0.0187 0.0176 0.0165 0.0156 0.0304 0.0284 0.0266 0.0251 0.0237 0.0410 0.0383 0.0360 0.0338 0.0320

0.005
0.010
0.020
0.025
0.030
0.040
0.050
0.075
0.100

0.0 0.0063 0.0125 0.0251 0.0313 0.0376 0.0502 0.0627 0.0941 0.1257
0.2 0.0054 0.0108 0.0216 0.0270 0.0324 0.0432 0.0541 0.0814 0.1090
0.4 0.0047 0.0094 0.0188 0.0235 0.0282 0.0377 0.0472 0.0712 0.0955
0.6 0.0041 0.0083 0.0165 0.0207 0.0249 0.0333 0.0417 0.0629 0.0845
0.8 0.0037 0.0073 0.0147 0.0184 0.0221 0.0296 0.0371 0.0561 0.0754
1.0 0.0033 0.0066 0.0132 0.0165 0.0199 0.0266 0.0333 0.0504 0.0679

For situations in which the truncation point is not precisely known, it can be estimated using order statistics or other approaches, as discussed in Aitchison and Silvey, Robson and Whitlock, and Sarhan and Greenberg.[2] [9]

where $f(t)$ is as defined in Equation 7.

$$\mu_t = E(t) = \int_0^\infty t f(t)\,dt \quad (8)$$

Defining $H(k)$ as

$$H(k) = \int_k^\infty z f(z)\,dz \quad (9)$$

where $f(t)$ is as defined in Equation 7.

$$\mu_t = E(t) = \int_0^\infty t f(t)\,dt \quad (8)$$

Defining $H(k)$ as

$$H(k) = \int_k^\infty z f(z)\,dz \quad (9)$$

Point of Truncation (k_L)

-2.0	-1.8	-1.6	-1.4	-1.2
0.0833	0.0580	0.0412	0.0301	0.0225
0.1550	0.1106	0.0800	0.0589	0.0444
0.2753	0.2037	0.1512	0.1135	0.0866
0.3272	0.2455	0.1843	0.1395	0.1070
0.3749	0.2847	0.2158	0.1646	0.1269
0.4605	0.3569	0.2751	0.2126	0.1656

[10] Then, Table 1 can be utilized.

Alternatively, the sample mean and standard deviation can be used to estimate the coefficient of variation as follows

$$c = \frac{\sigma}{\mu} \approx \frac{\sigma'}{x_i} \qquad (16)$$

where the accuracy of this approximation will improve as the sample size increases. This estimated coefficient of variation *c* can then be used along with Tables 1 and 2 to identify the approximate standardized truncation point k_L

EXTENSION TO THE RIGHT-TRUNCATED NORMAL DISTRIBUTION

In a fashion analogous to this paper's treatment of the left-truncated normal distribution, a "standardized, right-truncated normal distribution" may be defined using a stand ardizing variable $t = z - k_R$, which has the effect of defining the point of truncation as $t = 0$. The standardized, right-truncated normal distribution $f_{SRTN}(t)$ is, thus, given by

Truncation Point, kL $f(t + k_R)$

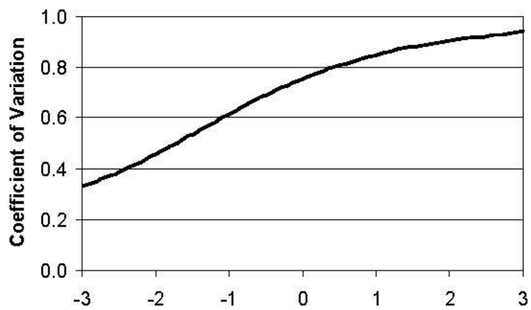


Figure 7 – Coefficient of Variation as a Function of *f* (*t*)

=

I

f (*z*)*dz*

SRTN

Truncation Point k_L for the Standardized, Left-Truncated

(17)

$-\infty$

Normal Distribution

0, *t* □ 0

Example of Use of Tables

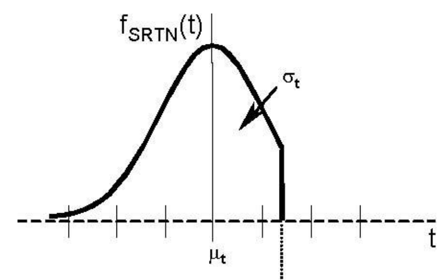
Figure 8.

An inventory manager is developing a plan for the stocking of an item whose monthly demand is normally distributed with a mean of 20 units and a standard deviation of 10 units. Clearly, monthly demand for this item cannot be less than 0 – so that this demand pattern would be most appropriately modeled as that of a left-truncated normal distribution. Using the notation of Figure 1: μ = 20, σ = 10, and $x_L = 0$. In terms of the

standard normal distribution, the truncation point k_L can be calculated using Equation 5

t

≤



−μ 0 −20

x_L

Figure 8 – Standardized, Right-Truncated Normal

k =

=−2=

L

Distribution

σ 10

a n d i s i l l u s t r a t e d i n

Suppose that the manager desires to insure that enough stock is on hand to satisfy 95% of the likely demand – i.e., to achieve a 95% service level. What stock level would be appropriate?

Using Table 1, for a truncation point of k_L = -2, to achieve *F(t)* = 0.95 requires a standardized *t* value of *t* = 3.6560. This, then, implies a standard normal value *z* of

$$z = t + k = 3.6560 - 2 = 1.6560$$

and a "real-world" stock level of

$$x = \mu + z\sigma = 20 + 1.6560(10) = 36.56$$

The symmetry between the standardized, right-truncated normal distribution and the standardized, left-truncated normal distribution is apparent *via* comparison of Figure 8 with Figure 3. This is further illustrated by comparing Figure 7 with Figure 9 – which graphs the coefficient of variation as a function of the truncation point k_R for the standardized, right-truncated normal distribution.

In other words, the manager should plan for an inventory level of 36.56 units in order to be able to satisfy 95% of the expected demands. This value can be contrasted with the one of 36.45 units that would be obtained using non-truncated distributions.

The difference in these values, although small for this example, illustrates the fact that Table 1 accounts for the fact that the mean of the truncated distribution is actually 20.552 units (slightly greater than the non-truncated distribution) and its standard deviation is actually 9.415 units (slightly less than the non-truncated case).

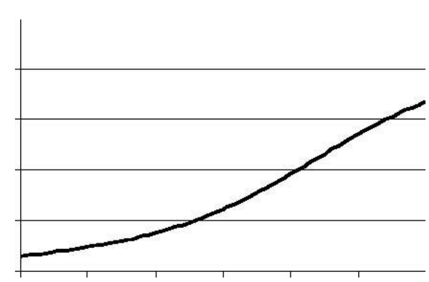


Figure 9 – Coefficient of Variation as a Function of Truncation Point k_R for the Standardized, Right-Truncated Normal Distribution

Clearly, Tables 1 and 2 can be used for problems involving the standardized, right-truncated normal distribution *via* the following reformulations:

- For a given value of k_R, use a value of $-k_L$ when using the tables.
- For a value of $F_{SRTN}(t)$, use a value of $1 - F_{SLTN}(t)$ when using the tables.
- The mean and coefficient of variation will be negative – by definition.

CONCLUSION

This paper has presented tables of the cumulative distribution function of the left-truncated normal distribution for use by workers/researchers in management science, finance, and the many other fields in which this family of distributions has found application. In the process of developing these tables, the characteristic parameters of this distribution were derived in terms of a "standardized" truncated distribution – for maximum generality. It was noted that the truncation point solely and uniquely determines the coefficient of variation – which, in turn, can be used to estimate the point of truncation.

ENDNOTES

Additional (and more extensive) reference tables of the cumulative distribution functions and characteristic parameters of the left-and right-truncated normal distribution are available for download at

http://domin.dom.edu/faculty/ajohnson/truncnorm.htm

REFERENCES

[1] Abramowitz, M., and Stegun, I., *Handbook of Mathematical Functions*, New York: Dover Publications, 1972.

[2] Aitchison, J., and Silvey, S.D., "Maximum-Likelihood Estimation of Parameters Subject to Restraints," *Annals of Mathematical Statistics*, 1958 **29**, 813-828.

[3] Barr, D.R., and Sherrill, E.T., "Mean and Variance of Truncated Normal Distributions," *The American Statistician*, 1999, **53**, 357-361.

[4] Cohen, A.C., *Truncated and Censored Samples*, New York: Marcel Dekker, 1991.

[5] Hald, A., *Statistical Theory with Engineering Applications*, New York: John Wiley and Sons, 1952.

[6] Johnson, A., "On the Truncated Normal Distribution: Characteristics of Singly- and Doubly-truncated Populations of Application in Management Science," Doctoral Dissertation, Stuart Graduate School of Business, Illinois Institute of Technology, Chicago, IL, 2001.

[7] Johnson, N.L., and Kotz, S., *Distributions in Statistics: Continuous Univariate Distributions 1*, New York: Houghton Mifflin, 1970.

[8] Kececioglu, D., *Reliability Engineering Handbook (Vol. 1)*, Englewood Cliffs: Prentice Hall, 1991.

[9] Robson, D.S., and Whitlock, J.H., "Estimation of a Truncation Point," *Biometrika*, 1964, **51**, 33-39.

[10] Sarhan, A.E., and Greenberg, B.G., "Estimation of Location and Scale Parameters by Order Statistics From Single and Doubly Censored Samples," *Annals of Mathematical Statistics*, 1956, **27**, 427
451.

[11] Schneider, H., *Truncated and Censored Samples From Normal Populations*, New York: Marcel Dekker, 1986.

[12] Thomopoulos, N.T., *Applied Forecasting Methods*, Englewood Cliffs: Prentice-Hall, 1980.