Use of the Left-Truncated Normal Distribution for Improving Achieved Service Levels

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Abstract: To determine the safety stock required to achieve a desired service level, demands are commonly assumed to follow a normal probability distribution. However, since demands are necessarily truncated at values below zero, this paper will show that this approach underestimates the actual demand level and results in achieved service levels that are less than those targeted. Use of a left-truncated normal probability distribution – i.e., a normal probability distribution in which values below a truncation point cannot be observed – for demand to determine safety stocks will be shown to improve the achieved service level. This paper describes a methodology for using standardized, left-truncated normal distributions to determine safety stocks, presents tables for use in implementing this methodology, and discusses the improvements possible through this approach.

INTRODUCTION

The normal – or Gaussian – distribution is one of the most widely utilized of all random variables. Mound-shaped or approximately mound-shaped distributions are encountered in a large number of applications and, via the Central Limit Theorem, provide the underpinning for the characteristics of sampling distributions upon which statistical inference is based. Despite this utility, the fact that the values of a normally distributed random variable can, in theory, assume any value over the range from -∞ to +∞ may lead to significant computational errors in applications where the distribution's outcomes are actually constrained. A case in point is the estimation of demand levels for determining safety stock required to achieve a desired service level.

This paper will review the limitations of the conventional methodology used to establish safety stocks, which utilizes non-truncated normal distributions. It will, then, describe and illustrate the use of an enhanced methodology to calculate safety stock based on a standardized, left-truncated normal distribution.

CONVENTIONAL METHODOLOGY FOR ESTABLISHING SAFETY STOCK

In the case of inventory management, demands are commonly assumed to follow a normal probability distribution when determining the safety stock required to achieve a desired service level for items that are subject to a continual review. As discussed in Thomopoulos (1990), this methodology places an order for a specified order quantity $SS = z_0 \sigma_L$ (Q) when the on-hand plus on-order inventory falls below a pre-determined order point (OP) level. Brown (1962) has shown that, if the demand over the lead time (L) is assumed to be normally distributed with mean μ_L and standard deviation σ_L, the safety stock (SS) and order point for a desired service level (SL) can be found using statistical $OP = \mu_L + SS$ methods. (In terms of the forecast, μ_L is the forecast demand for the lead time period and σ_L is the standard deviation of the forecast errors over the same time period.) In particular, the safety stock is given by

(1)

and the order point is given by

$$E(\text{pieces short}) = \sigma_L E(z > z_0)$$

(2)

The value of z_0 in Equation 1 is called the "safety factor" and is calculated in order to achieve the desired service level by recognizing that the expected number of pieces short – i.e., the expected value of the demand over the lead time that is greater than the order point – is given by

$$SL = 1 - \frac{E(z > z_0)\sigma_L}{Q}$$

(3)

where $E(z > z_0)$ gives the expected value of the standard normal variate (z) beyond the value z_0 and is commonly called the "partial expectation of z beyond z_0." Using Equation 3, the service level, which is defined as the demand filled divided by total demand, can be shown to be

(4)

Equation 4 can be rearranged for computational use as the following:

$$E(z > z_0) = \frac{(1 - SL)Q}{\sigma_L} \tag{5}$$

Equation 5 provides a means of determining the value of the safety factor z_0 necessary to achieve a service level for a given demand pattern and order quantity.

Such methods are only as good as their underlying assumptions. Agrawal and Smith (1996) note the under-performance of traditional demand pattern models (e.g., non-truncated normal and Poisson) in achieving a desired service level. They argue that the negative binomial distribution is a better model to employ. Thomopoulos (1980) illustrates the use of left-truncated normal distributions to determine safety stocks. Specifically, by recognizing that the impossibility of negative demands effectively truncates otherwise normally-distributed demand patterns, he shows how the expected value of the truncated normal distribution can be used to more accurately estimate the safety stock required to achieve a desired service level. This paper builds upon his work as well as upon that of Johnson (2001) in order to develop an enhanced methodology for establishing safety stocks.

Additional applications of the truncated normal distribution to inventory problems can be found in Bookbinder and Lordahl (1989) and Sinha (1991) – who utilize truncated normal distributions (among other distribution types) to evaluate the performance of inventory re-ordering systems operating under stochastic lead time and demand patterns. Bookbinder and Lordahl (1989) use the left-truncated normal distribution (at three levels of truncation) as one approach to simulate the stochastic nature of demand patterns in order to compare the performance of their "bootstrap statistical procedure" against traditional methods to set the re-order point. The left-truncated normal distribution is one of the distributions used by Sinha (1991) to simulate stochastic demand patterns for purposes of establishing an optimal policy for a continuous review inventory system.

LEFT-TRUNCATED NORMAL DISTRIBUTION

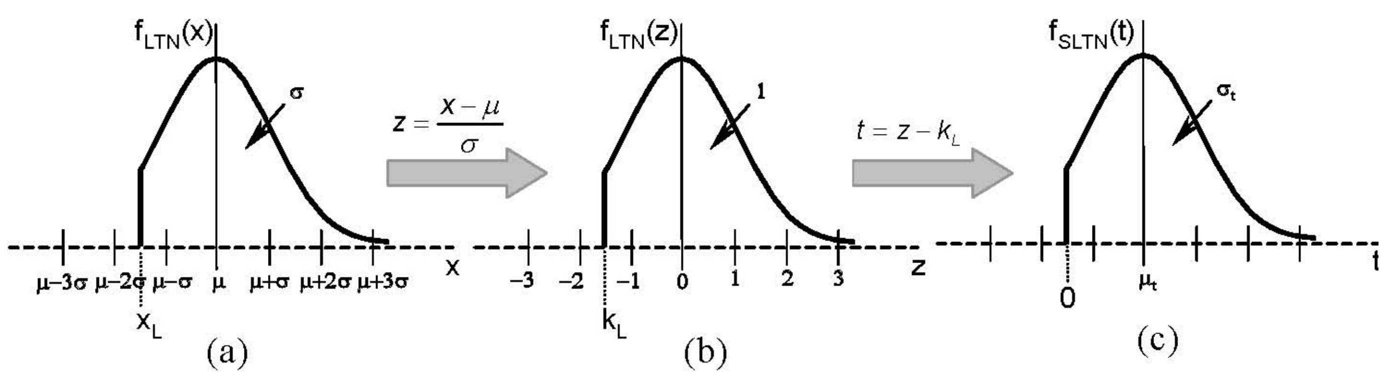
Definition

Consider a normally-distributed random variable x with a probability density function f(x) specified as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty \le x \le \infty \tag{6}$$

If the values of x below some value x_L cannot be observed – due to censoring or truncation – then, as shown in Figure 1a and following Hald's (1952) conventions, the resulting distribution is a left-truncated normal distribution with probability density function $f_{LTN}(x)$ given by

$$f_{LTN}(x) = \begin{cases} 0, & -\infty \le x \le x_L \\ \dfrac{f(x)}{\int_{x_L}^{\infty} f(x)dx}, & x_L \le x \le \infty \end{cases} \tag{7}$$



where f(x) is as defined in Equation 6.

Figure 1: Left-Truncated Normal Distribution

$$z = \frac{x - \mu}{\sigma} \tag{8}$$

For purposes of generality, Equation 6 can be re-stated in terms of the standard normal distribution (denoted f(z)) where and

$$f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}, -\infty \le z \le \infty \quad (9)$$

The point of truncation x_L can also be expressed in terms of the standard normal distribution as given by

$$k_L=\frac{x_L-\mu}{\sigma} \quad (10)$$

Reformulating the left-truncated normal distribution of Equation 7 in terms of the standard normal distribution, the following can be found:

$$f_{LTN}(z)=\begin{cases}0, & -\infty \le z \le k_L \\ \dfrac{f(z)}{\int_{k_L}^{\infty} f(z)dz}, & k_L \le z \le \infty\end{cases} \quad (11)$$

This is illustrated in Figure 1b. Similar expressions are found in Johnson and Kotz (1970), Schneider (1986), and Cohen (1991).

To define what Thomopoulos (1980) terms as a "standardized, left-truncated normal distribution," a standardizing variable t = z – k_L is introduced, which has the effect of defining the point of truncation as t = 0. The standardized, left-truncated normal distribution $f_{SLTN}(t)$ is, thus, given by

$$f_{SLTN}(t)=\begin{cases}0, & t \le 0 \\ \dfrac{f(t+k_L)}{\int_{k_L}^{\infty} f(z)dz}, & t \ge 0\end{cases} \quad (12)$$

The standardized, left-truncated normal distribution is illustrated in Figure 1c.

Parameters

Consider the mean (μ_t) of the standardized, left-truncated normal distribution – for a given point of truncation k_L, where $f_{SLTN}(t)$ is as defined in Equation 12.

$$\mu_t = E(t)=\int_0^{\infty} t f_{SLTN}(t)dt \quad (13)$$

Given t = z – k_L, it follows that dt = dz, z = k_L for t = 0, and z = ∞ for t = ∞. Equation 13 can, then, be reformulated as

$$\mu_t=\frac{1}{\int_{k_L}^{\infty} f(z)dz}\int_{k_L}^{\infty}(z-k_L)f(z)dz = \frac{1}{\int_{k_L}^{\infty} f(z)dz}\left[\int_{k_L}^{\infty} zf(z)dz - k_L\int_{k_L}^{\infty} f(z)dz\right] \quad (14)$$

Defining H(k) as

$$H(k)=\int_k^{\infty} f(z)dz \quad (15)$$

and performing the integrations indicated, Equation 13 can be shown to result in

$$\mu_t=\frac{1}{H(k_L)}\left[f(k_L)-k_L H(k_L)\right] \quad (16)$$

From Equation 16, it is clear that the mean of the standardized, left-truncated normal distribution is uniquely determined by and solely dependent upon the point of truncation, k_L.

Consider now the standard deviation (σ_t) of the standardized, left-truncated normal distribution. Given that

$$\sigma_t^2 = E(t^2) - \mu_t^2 \qquad (17)$$

It is only necessary to calculate $E(t^2)$.

$$E(t^2) = \int_0^\infty t^2 f(t)dt = \frac{1}{H(k_L)} \int_{k_L}^\infty \left(z - k_L\right)^2 f(z)dz$$

(18)

$$= \frac{1}{H(k_L)} \left[\int_{k_L}^\infty z^2 f(z)dz - 2k_L \int_{k_L}^\infty zf(z)dz + k_L^2 \int_{k_L}^\infty f(z)dz \right]$$

Performing the indicated integrations, Equation 18 can be shown to result in

$$E(t^2) = \frac{1}{H(k_L)} \left[(1 + k_L^2)H(k_L) - k_L f(k_L) \right] \quad (19)$$

Equation 19 can be used along with Equation 16 to calculate σ_t as shown in Equation 17. Once again, it is worth noting that the point of truncation k_L uniquely and solely determines the standard deviation of the standardized, left-truncated normal distribution.

To facilitate subsequent calculations, a coefficient of variation c – which, again, exists uniquely for a particular k_L – can be defined as

$$c = \frac{\sigma_t}{\mu_t} \quad (20)$$

For demands truncated at zero, this coefficient of variation is approximately equal to 0.33 when $k_L = -3$ ("light truncation") and approaches 1.0 when $k_L > 3$ ("heavy truncation").

The expressions derived in this section for the mean and standard deviation of the standardized, left-truncated normal distribution can be shown to be equivalent to those presented in Schneider (1986). In particular, they represent a re-derivation of the work presented in Thomopoulos (1980). Barr and Sherrill (1999) have published an expression for the variance of the left-truncated normal distribution in terms of the Chi-Square distribution. Johnson (2001) includes tables and quantitative visualizations of the cumulative distribution functions of standardized, left-truncated normal distributions as a function of the point of truncation.

ENHANCED METHODOLOGY FOR ESTABLISHING SAFETY STOCK LEVELS

Explanation of Method

If demands are assumed to follow a left-truncated normal probability distribution, then, as noted earlier, the truncation point k_L uniquely determines the distribution's mean μ_t and standard deviation σ_t (i.e., its coefficient of variation). Introducing a standardized truncated variable w where

$$w = \frac{t - \mu_t}{\sigma_t} \quad (21)$$

Equation 5 can now be restated (more accurately) as

$$E(w > w_0) = \frac{(1 - SL)Q}{\sigma_L}$$

$$E(w > w_0) = \frac{E(t > t_0)}{\sigma_t} \quad (22)$$

where $E(w > w_0)$ is the expected value of w greater than some specified value w_0, the safety factor. Once the value of w_0 that satisfies Equation 22 is found, the safety stock can be calculated using a reformulation of Equation 1

$$SS = w_0 \sigma_L \quad (23)$$

Calculation of $E(w > w_0)$ in terms of $E(t > t_0)$ follows from Equation 21 with

(24)

Similarly, $E(t > t_0)$ – for a particular truncation point k_L – can be determined using

$$E(t > t_0)_{k_L} = \frac{E(z > z_0)}{H(k_L)} \quad (25)$$

Combining Equations 24 and 25 results in

$$E(w > w_0) = \frac{E(z > z_0)}{H(k_L)\sigma_t} \qquad (26)$$

with

$$z_0 = \mu_t + w_0\sigma_t + k_L \qquad (27)$$

Thus, for a particular demand pattern – defined by the mean and standard deviation of the demand (i.e., a truncation point) – a value of the safety factor w_0 can be found that satisfies Equation 22, given a targeted service level and order quantity.

Development of Tables

To facilitate the implementation of this methodology, this paper now presents tables for use in calculating the safety factor w_0 for a particular situation. Table 1 lists the value of w_0 associated with a particular service level as a function of the coefficient of variation (σ_t/μ_t) and ratio of the order quantity to the standard deviation of demand (Q/σ_L). Table 1 is presented in three segments for service levels of 90%, 95%, and 99%, respectively – with values of $w_0 < 0$ suppressed (i.e., no safety stock required).

Q/σ_L Ratio	Coefficient of Variation (σ_t/μ_t) 0.20	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
1.0	0.902	0.905	0.912	0.925	0.944	0.968	0.997	1.030	1.065	1.102	1.140	1.178	1.215	1.250	1.255	1.181
1.1	0.850	0.852	0.859	0.871	0.889	0.912	0.940	0.970	1.002	1.036	1.070	1.104	1.136	1.166	1.167	1.095
1.2	0.801	0.803	0.810	0.821	0.839	0.860	0.886	0.914	0.944	0.975	1.006	1.036	1.064	1.089	1.087	1.016
1.3	0.755	0.757	0.763	0.775	0.791	0.812	0.835	0.862	0.889	0.918	0.946	0.972	0.997	1.017	1.013	0.944
1.4	0.712	0.714	0.720	0.731	0.746	0.766	0.788	0.813	0.838	0.864	0.889	0.913	0.934	0.951	0.944	0.876
1.5	0.671	0.673	0.679	0.689	0.704	0.722	0.743	0.766	0.790	0.813	0.836	0.857	0.875	0.889	0.879	0.814
1.6	0.632	0.634	0.640	0.649	0.663	0.681	0.701	0.722	0.744	0.765	0.786	0.804	0.819	0.830	0.819	0.755
1.7	0.595	0.597	0.602	0.612	0.625	0.641	0.660	0.680	0.700	0.720	0.738	0.754	0.767	0.775	0.762	0.699
1.8	0.560	0.562	0.566	0.575	0.588	0.604	0.621	0.640	0.658	0.676	0.692	0.706	0.716	0.722	0.708	0.646
1.9	0.526	0.527	0.532	0.541	0.553	0.568	0.584	0.601	0.619	0.635	0.649	0.661	0.669	0.673	0.657	0.596
2.0	0.493	0.495	0.499	0.507	0.519	0.533	0.548	0.564	0.580	0.595	0.607	0.617	0.623	0.625	0.608	0.549
2.1	0.461	0.463	0.467	0.475	0.486	0.500	0.514	0.529	0.543	0.556	0.567	0.575	0.580	0.580	0.561	0.504
2.2	0.431	0.432	0.437	0.444	0.455	0.467	0.481	0.495	0.508	0.520	0.529	0.535	0.538	0.536	0.517	0.461
2.3	0.401	0.403	0.407	0.414	0.424	0.436	0.449	0.462	0.474	0.484	0.492	0.497	0.498	0.494	0.474	0.419
2.4	0.373	0.374	0.378	0.385	0.395	0.406	0.418	0.430	0.440	0.449	0.456	0.460	0.459	0.454	0.433	0.380
2.5	0.345	0.346	0.350	0.357	0.366	0.376	0.388	0.398	0.408	0.416	0.422	0.424	0.422	0.416	0.394	0.341
2.6	0.318	0.319	0.323	0.329	0.338	0.348	0.358	0.368	0.377	0.384	0.388	0.389	0.386	0.378	0.356	0.305
2.7	0.292	0.293	0.296	0.303	0.311	0.320	0.330	0.339	0.347	0.353	0.356	0.355	0.351	0.342	0.320	0.269
2.8	0.266	0.267	0.271	0.276	0.284	0.293	0.302	0.311	0.317	0.322	0.324	0.323	0.317	0.308	0.285	0.235
2.9	0.241	0.242	0.245	0.251	0.258	0.267	0.275	0.283	0.289	0.292	0.293	0.291	0.285	0.274	0.250	0.202
3.0	0.217	0.218	0.221	0.226	0.233	0.241	0.249	0.256	0.261	0.264	0.264	0.260	0.253	0.241	0.218	0.170
3.1	0.193	0.194	0.197	0.202	0.209	0.216	0.223	0.229	0.234	0.236	0.235	0.230	0.222	0.210	0.186	0.139
3.2	0.169	0.170	0.173	0.178	0.185	0.191	0.198	0.203	0.207	0.208	0.206	0.201	0.192	0.179	0.155	0.109
3.3	0.146	0.148	0.150	0.155	0.161	0.167	0.173	0.178	0.181	0.181	0.179	0.173	0.163	0.149	0.124	0.080
3.4	0.124	0.125	0.128	0.132	0.138	0.144	0.149	0.153	0.155	0.155	0.152	0.145	0.134	0.120	0.095	0.052
3.5	0.102	0.103	0.106	0.110	0.115	0.121	0.126	0.129	0.131	0.129	0.125	0.118	0.107	0.092	0.067	0.024
3.6	0.080	0.081	0.084	0.088	0.093	0.098	0.103	0.105	0.106	0.104	0.100	0.091	0.080	0.064	0.039	***
3.7	0.059	0.060	0.063	0.066	0.071	0.076	0.080	0.082	0.082	0.080	0.074	0.065	0.053	0.037	0.012	***
3.8	0.038	0.039	0.042	0.045	0.050	0.054	0.058	0.059	0.059	0.056	0.050	0.040	0.027	0.011	***	***
3.9	0.018	0.019	0.021	0.025	0.029	0.033	0.036	0.037	0.036	0.032	0.025	0.015	0.002	***	***	***
4.0	***	***	0.001	0.004	0.008	0.012	0.014	0.015	0.013	0.009	0.002	***	***	***	***	***

Table 1.1: Safety Factor w_0 as a Function of the Coefficient of Variation and Q/σ_L Ratio for a 90% Service Level

Use of Tables

To illustrate the use of Table 1, consider an inventory manager who needs to determine the safety stock necessary to achieve a 95% service level for a product whose average lead-time demand (μ_L) is 50 units – with a standard deviation (σ_L) of 40 units. The order quantity (Q) that has been established for this item is 80 units.

Given the values above, the coefficient of variation (c) can be calculated – using Equation 20 – to be 0.8. Similarly, the ratio Q/σ_L is seen to be 2.0. From Table 1.2, the value of w_0 associated with this c and Q/σ_L can be found to be 1.178. The required safety stock can be calculated to be 47-48 units using Equation 23. This value can be compared to the safety stock value of 36 units that would have been obtained using traditional methods and Equation 5.

If the order quantity (Q) were 50 units, then the ratio Q/σ_L would be 1.25, and the associated value of the safety factor w_0 could be interpolated from those in Table 1.2 to be approximately 1.5. This would result in a safety stock of 60 units. This is, again, greater than the value of 46 units that would be obtained using non-truncated methods.

Q/σ_L Ratio	Coefficient of Variation (σ_t/μ_t)															
	0.20	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
1.0	1.256	1.259	1.268	1.285	1.312	1.347	1.389	1.438	1.493	1.554	1.619	1.689	1.763	1.840	1.879	1.793
1.2	1.167	1.170	1.179	1.195	1.219	1.252	1.290	1.335	1.385	1.439	1.498	1.559	1.623	1.688	1.717	1.634
1.4	1.090	1.093	1.101	1.116	1.139	1.169	1.205	1.246	1.291	1.340	1.392	1.446	1.502	1.557	1.579	1.498
1.6	1.021	1.024	1.032	1.046	1.068	1.095	1.129	1.167	1.208	1.253	1.299	1.347	1.396	1.443	1.458	1.380
1.8	0.959	0.962	0.969	0.983	1.003	1.029	1.060	1.095	1.133	1.174	1.216	1.259	1.301	1.341	1.351	1.275
2.0	0.902	0.905	0.912	0.925	0.944	0.968	0.997	1.030	1.065	1.102	1.140	1.178	1.215	1.250	1.255	1.181
2.2	0.850	0.852	0.859	0.871	0.889	0.912	0.940	0.970	1.002	1.036	1.070	1.104	1.136	1.166	1.167	1.095
2.4	0.801	0.803	0.810	0.821	0.839	0.860	0.886	0.914	0.944	0.975	1.006	1.036	1.064	1.089	1.087	1.016
2.6	0.755	0.757	0.763	0.775	0.791	0.812	0.835	0.862	0.889	0.918	0.946	0.972	0.997	1.017	1.013	0.944
2.8	0.712	0.714	0.720	0.731	0.746	0.766	0.788	0.813	0.838	0.864	0.889	0.913	0.934	0.951	0.944	0.876
3.0	0.671	0.673	0.679	0.689	0.704	0.722	0.743	0.766	0.790	0.813	0.836	0.857	0.875	0.889	0.879	0.814
3.2	0.632	0.634	0.640	0.649	0.663	0.681	0.701	0.722	0.744	0.765	0.786	0.804	0.819	0.830	0.819	0.755
3.4	0.595	0.597	0.602	0.612	0.625	0.641	0.660	0.680	0.700	0.720	0.738	0.754	0.767	0.775	0.762	0.699
3.6	0.560	0.562	0.566	0.575	0.588	0.604	0.621	0.640	0.658	0.676	0.692	0.706	0.716	0.722	0.708	0.646
3.8	0.526	0.527	0.532	0.541	0.553	0.568	0.584	0.601	0.619	0.635	0.649	0.661	0.669	0.673	0.657	0.596
4.0	0.493	0.495	0.499	0.507	0.519	0.533	0.548	0.564	0.580	0.595	0.607	0.617	0.623	0.625	0.608	0.549
4.2	0.461	0.463	0.467	0.475	0.486	0.500	0.514	0.529	0.543	0.556	0.567	0.575	0.580	0.580	0.561	0.504
4.4	0.431	0.432	0.437	0.444	0.455	0.467	0.481	0.495	0.508	0.520	0.529	0.535	0.538	0.536	0.517	0.461
4.6	0.401	0.403	0.407	0.414	0.424	0.436	0.449	0.462	0.474	0.484	0.492	0.497	0.498	0.494	0.474	0.419
4.8	0.373	0.374	0.378	0.385	0.395	0.406	0.418	0.430	0.440	0.449	0.456	0.460	0.459	0.454	0.433	0.380
5.0	0.345	0.346	0.350	0.357	0.366	0.376	0.388	0.398	0.408	0.416	0.422	0.424	0.422	0.416	0.394	0.341
5.2	0.318	0.319	0.323	0.329	0.338	0.348	0.358	0.368	0.377	0.384	0.388	0.389	0.386	0.378	0.356	0.305
5.4	0.292	0.293	0.296	0.303	0.311	0.320	0.330	0.339	0.347	0.353	0.356	0.355	0.351	0.342	0.320	0.269
5.6	0.266	0.267	0.271	0.276	0.284	0.293	0.302	0.311	0.317	0.322	0.324	0.323	0.317	0.308	0.285	0.235
5.8	0.241	0.242	0.245	0.251	0.258	0.267	0.275	0.283	0.289	0.292	0.293	0.291	0.285	0.274	0.250	0.202
6.0	0.217	0.218	0.221	0.226	0.233	0.241	0.249	0.256	0.261	0.264	0.264	0.260	0.253	0.241	0.218	0.170
6.2	0.193	0.194	0.197	0.202	0.209	0.216	0.223	0.229	0.234	0.236	0.235	0.230	0.222	0.210	0.186	0.139
6.4	0.169	0.170	0.173	0.178	0.185	0.191	0.198	0.203	0.207	0.208	0.206	0.201	0.192	0.179	0.155	0.109
6.6	0.146	0.148	0.150	0.155	0.161	0.167	0.173	0.178	0.181	0.181	0.179	0.173	0.163	0.149	0.124	0.080
6.8	0.124	0.125	0.128	0.132	0.138	0.144	0.149	0.153	0.155	0.155	0.152	0.145	0.134	0.120	0.095	0.052
7.0	0.102	0.103	0.106	0.110	0.115	0.121	0.126	0.129	0.131	0.129	0.125	0.118	0.107	0.092	0.067	0.024
7.2	0.080	0.081	0.084	0.088	0.093	0.098	0.103	0.105	0.106	0.104	0.100	0.091	0.080	0.064	0.039	***
7.4	0.059	0.060	0.063	0.066	0.071	0.076	0.080	0.082	0.082	0.080	0.074	0.065	0.053	0.037	0.012	***
7.6	0.038	0.039	0.042	0.045	0.050	0.054	0.058	0.059	0.059	0.056	0.050	0.040	0.027	0.011	***	***
7.8	0.018	0.019	0.021	0.025	0.029	0.033	0.036	0.037	0.036	0.032	0.025	0.015	0.002	***	***	***
8.0	***	***	0.001	0.004	0.008	0.012	0.014	0.015	0.013	0.009	0.002	***	***	***	***	***
8.2	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***
8.4	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***
8.6	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***
8.8	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***
9.0	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***

Table 1.2: Safety Factor w_0 as a Function of the Coefficient of Variation and Q/σ_L Ratio for a 95% Service Level

ILLUSTRATIONS OF ENHANCEMENT OF SERVICE LEVEL

While the examples used to illustrate the functionality of Table 1 provided some insight as to the benefit of the use of left-truncated normal distribution as the model for demand versus the use of conventional methods, this section presents tables for use in estimating the magnitude of the errors introduced by not utilizing the improved methodology described in this paper.

Table 2 lists the ratio of the actual (achieved) to targeted (desired) service level obtained using conventional (non-truncated) methods as a function of the targeted service level and the coefficient of variation of the demand – with ratios < 0 suppressed.

$$(28)\quad Ratio = \frac{SL_{\text{achieved}}}{SL_{\text{targeted}}}$$

For example, continuing the example of the previous section, with a targeted service level of 95% and a coefficient of variation of 0.8, this ratio is 0.761. In other words, the achieved service level (using conventional, non-truncated demand assumptions) would be approximately 72% - i.e., 0.761(95%). As would be expected, this table shows that, as the coefficient of variation (and, correspondingly, the degree of truncation) increases, the achieved service level declines. This decline begins at lower levels of the coefficient of variation for lower targeted service levels – due to the truncation's greater effect at these levels. This behavior is illustrated graphically in contour plot form in Figure 2.

Table 3 provides a different perspective on the benefit of the enhanced methodology described in this paper. In this table, the coefficient of variation c at which a ratio of actual to targeted service level is achieved is shown as a function of the targeted service level. For example, assume one is targeting a service level of 95% but must insure that it is at least 90% (even in the presence of demand truncation). For this ratio of approximately 0.95, one cannot use traditional methods of establishing the safety stock for item demands whose coefficients of variation exceed 0.679. Figure 3 provides a contour plot, which illustrates the behavior of this "critical" coefficient of variation as a function of the targeted service level and acceptable ratio of actual to targeted performance.

Q/σ_L Ratio	Coefficient of Variation (σ_t/μ_t) 0.20	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
1.0	1.938	1.943	1.957	1.984	2.025	2.082	2.153	2.239	2.339	2.454	2.585	2.735	2.906	3.102	3.250	3.150
1.2	1.868	1.873	1.886	1.912	1.952	2.006	2.074	2.156	2.251	2.360	2.484	2.625	2.784	2.966	3.100	3.001
1.4	1.808	1.812	1.825	1.850	1.889	1.941	2.006	2.084	2.175	2.279	2.397	2.530	2.680	2.849	2.972	2.874
1.6	1.754	1.759	1.771	1.795	1.833	1.883	1.946	2.021	2.109	2.208	2.321	2.447	2.588	2.747	2.860	2.763
1.8	1.707	1.711	1.723	1.746	1.783	1.832	1.893	1.965	2.049	2.145	2.252	2.372	2.507	2.657	2.761	2.665
2.0	1.663	1.667	1.679	1.702	1.737	1.785	1.844	1.914	1.995	2.087	2.190	2.305	2.433	2.575	2.672	2.576
2.2	1.623	1.627	1.639	1.661	1.696	1.742	1.799	1.867	1.946	2.034	2.134	2.244	2.366	2.501	2.591	2.496
2.4	1.586	1.590	1.602	1.624	1.657	1.702	1.758	1.824	1.900	1.986	2.081	2.187	2.304	2.433	2.517	2.423
2.6	1.552	1.556	1.567	1.588	1.621	1.665	1.720	1.784	1.858	1.941	2.033	2.135	2.247	2.369	2.449	2.355
2.8	1.520	1.524	1.535	1.556	1.588	1.631	1.684	1.746	1.818	1.898	1.988	2.086	2.193	2.311	2.385	2.292
3.0	1.490	1.493	1.504	1.525	1.556	1.598	1.650	1.711	1.780	1.859	1.945	2.040	2.143	2.256	2.326	2.233
3.2	1.461	1.465	1.475	1.496	1.526	1.567	1.618	1.677	1.745	1.821	1.905	1.997	2.096	2.204	2.270	2.178
3.4	1.434	1.438	1.448	1.468	1.498	1.538	1.588	1.646	1.712	1.786	1.867	1.956	2.052	2.155	2.217	2.126
3.6	1.408	1.412	1.422	1.442	1.471	1.511	1.559	1.616	1.680	1.752	1.831	1.917	2.010	2.109	2.167	2.077
3.8	1.384	1.387	1.397	1.416	1.446	1.484	1.532	1.587	1.650	1.720	1.797	1.880	1.970	2.065	2.120	2.031
4.0	1.360	1.364	1.373	1.392	1.421	1.459	1.505	1.560	1.621	1.689	1.764	1.845	1.931	2.023	2.075	1.987
4.2	1.338	1.341	1.351	1.369	1.397	1.435	1.480	1.533	1.593	1.660	1.733	1.811	1.895	1.983	2.033	1.944
4.4	1.316	1.319	1.329	1.347	1.375	1.411	1.456	1.508	1.567	1.632	1.703	1.779	1.860	1.945	1.992	1.904
4.6	1.295	1.298	1.308	1.326	1.353	1.389	1.433	1.484	1.541	1.605	1.674	1.748	1.826	1.909	1.953	1.866
4.8	1.275	1.278	1.287	1.305	1.332	1.367	1.410	1.460	1.517	1.579	1.646	1.718	1.794	1.874	1.915	1.829
5.0	1.256	1.259	1.268	1.285	1.312	1.347	1.389	1.438	1.493	1.554	1.619	1.689	1.763	1.840	1.879	1.793
5.2	1.237	1.240	1.249	1.266	1.292	1.326	1.368	1.416	1.470	1.529	1.593	1.661	1.733	1.807	1.844	1.759
5.4	1.219	1.222	1.231	1.248	1.273	1.307	1.348	1.395	1.448	1.506	1.568	1.634	1.704	1.776	1.811	1.726
5.6	1.201	1.204	1.213	1.230	1.255	1.288	1.328	1.374	1.426	1.483	1.544	1.608	1.676	1.745	1.779	1.694
5.8	1.184	1.187	1.195	1.212	1.237	1.269	1.309	1.355	1.405	1.461	1.520	1.583	1.649	1.716	1.747	1.664
6.0	1.167	1.170	1.179	1.195	1.219	1.252	1.290	1.335	1.385	1.439	1.498	1.559	1.623	1.688	1.717	1.634
6.2	1.151	1.154	1.162	1.178	1.203	1.234	1.272	1.316	1.365	1.419	1.475	1.535	1.597	1.660	1.688	1.605
6.4	1.135	1.138	1.146	1.162	1.186	1.217	1.255	1.298	1.346	1.398	1.454	1.512	1.572	1.633	1.659	1.577
6.6	1.120	1.123	1.131	1.147	1.170	1.201	1.238	1.280	1.327	1.378	1.433	1.490	1.548	1.607	1.632	1.550
6.8	1.105	1.108	1.116	1.131	1.154	1.185	1.221	1.263	1.309	1.359	1.412	1.468	1.525	1.582	1.605	1.524
7.0	1.090	1.093	1.101	1.116	1.139	1.169	1.205	1.246	1.291	1.340	1.392	1.446	1.502	1.557	1.579	1.498
7.2	1.076	1.078	1.086	1.102	1.124	1.154	1.189	1.229	1.274	1.322	1.373	1.426	1.480	1.533	1.554	1.473
7.4	1.062	1.064	1.072	1.087	1.110	1.139	1.173	1.213	1.257	1.304	1.354	1.405	1.458	1.510	1.529	1.449
7.6	1.048	1.051	1.058	1.073	1.095	1.124	1.158	1.197	1.240	1.287	1.335	1.386	1.437	1.487	1.505	1.425
7.8	1.034	1.037	1.045	1.060	1.081	1.110	1.143	1.182	1.224	1.270	1.317	1.366	1.416	1.465	1.481	1.402
8.0	1.021	1.024	1.032	1.046	1.068	1.095	1.129	1.167	1.208	1.253	1.299	1.347	1.396	1.443	1.458	1.380
8.2	1.008	1.011	1.019	1.033	1.054	1.082	1.115	1.152	1.193	1.236	1.282	1.329	1.376	1.422	1.436	1.358
8.4	0.996	0.998	1.006	1.020	1.041	1.068	1.101	1.137	1.177	1.220	1.265	1.311	1.357	1.401	1.414	1.336
8.6	0.983	0.986	0.993	1.007	1.028	1.055	1.087	1.123	1.163	1.205	1.248	1.293	1.338	1.381	1.393	1.315
8.8	0.971	0.974	0.981	0.995	1.015	1.042	1.073	1.109	1.148	1.189	1.232	1.276	1.319	1.361	1.372	1.295
9.0	0.959	0.962	0.969	0.983	1.003	1.029	1.060	1.095	1.133	1.174	1.216	1.259	1.301	1.341	1.351	1.275
9.2	0.947	0.950	0.957	0.971	0.991	1.017	1.047	1.082	1.119	1.159	1.200	1.242	1.283	1.322	1.331	1.255
9.4	0.936	0.938	0.946	0.959	0.979	1.004	1.034	1.068	1.105	1.144	1.185	1.226	1.266	1.304	1.312	1.236
9.6	0.925	0.927	0.934	0.947	0.967	0.992	1.022	1.055	1.092	1.130	1.170	1.209	1.248	1.285	1.292	1.217
9.8	0.913	0.916	0.923	0.936	0.955	0.980	1.010	1.043	1.078	1.116	1.155	1.194	1.232	1.267	1.273	1.199
10.0	0.902	0.905	0.912	0.925	0.944	0.968	0.997	1.030	1.065	1.102	1.140	1.178	1.215	1.250	1.255	1.181

Table 1.3: Safety Factor w_0 as a Function of the Coefficient of Variation and Q/σ_L Ratio for a 99% Service Level

Targeted Service Level	Coefficient of Variation (σ_t/μ_t) 0.20	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
70%	1.000	0.999	0.994	0.984	0.967	0.937	0.891	0.819	0.700	0.492	0.076	***	***	***	***	***
71%	1.000	0.999	0.995	0.985	0.968	0.940	0.896	0.827	0.714	0.515	0.120	***	***	***	***	***
72%	1.000	0.999	0.995	0.986	0.970	0.943	0.901	0.836	0.728	0.539	0.162	***	***	***	***	***
73%	1.000	0.999	0.995	0.987	0.971	0.946	0.906	0.844	0.741	0.561	0.203	***	***	***	***	***
74%	1.000	0.999	0.995	0.987	0.973	0.949	0.911	0.851	0.754	0.583	0.243	***	***	***	***	***
75%	1.000	0.999	0.996	0.988	0.974	0.951	0.916	0.859	0.767	0.605	0.282	***	***	***	***	***
76%	1.000	0.999	0.996	0.989	0.975	0.954	0.920	0.866	0.779	0.625	0.319	***	***	***	***	***
77%	1.000	0.999	0.996	0.989	0.977	0.956	0.924	0.874	0.791	0.646	0.356	***	***	***	***	***
78%	1.000	0.999	0.996	0.990	0.978	0.959	0.929	0.881	0.803	0.665	0.392	***	***	***	***	***
79%	1.000	0.999	0.996	0.990	0.979	0.961	0.933	0.888	0.814	0.685	0.427	***	***	***	***	***
80%	1.000	0.999	0.997	0.991	0.981	0.963	0.937	0.894	0.825	0.703	0.461	***	***	***	***	***
81%	1.000	0.999	0.997	0.992	0.982	0.966	0.941	0.901	0.836	0.722	0.495	***	***	***	***	***
82%	1.000	0.999	0.997	0.992	0.983	0.968	0.944	0.907	0.846	0.740	0.527	0.004	***	***	***	***
83%	1.000	0.999	0.997	0.993	0.984	0.970	0.948	0.913	0.857	0.757	0.559	0.071	***	***	***	***
84%	1.000	0.999	0.997	0.993	0.985	0.972	0.952	0.919	0.867	0.774	0.590	0.136	***	***	***	***
85%	1.000	0.999	0.998	0.994	0.986	0.974	0.955	0.925	0.877	0.791	0.620	0.199	***	***	***	***
86%	1.000	0.999	0.998	0.994	0.987	0.976	0.959	0.931	0.886	0.807	0.649	0.261	***	***	***	***
87%	1.000	1.000	0.998	0.995	0.988	0.978	0.962	0.937	0.895	0.823	0.678	0.322	***	***	***	***
88%	1.000	1.000	0.998	0.995	0.989	0.980	0.965	0.942	0.905	0.838	0.706	0.381	***	***	***	***
89%	1.000	1.000	0.998	0.996	0.990	0.982	0.969	0.948	0.914	0.853	0.734	0.439	***	***	***	***
90%	1.000	1.000	0.999	0.996	0.991	0.984	0.972	0.953	0.922	0.868	0.761	0.496	***	***	***	***
91%	1.000	1.000	0.999	0.996	0.992	0.986	0.975	0.958	0.931	0.883	0.787	0.551	***	***	***	***
92%	1.000	1.000	0.999	0.997	0.993	0.987	0.978	0.963	0.939	0.897	0.813	0.605	***	***	***	***
93%	1.000	1.000	0.999	0.997	0.994	0.989	0.981	0.968	0.947	0.911	0.838	0.658	0.004	***	***	***
94%	1.000	1.000	0.999	0.998	0.995	0.991	0.984	0.973	0.955	0.924	0.862	0.710	0.155	***	***	***
95%	1.000	1.000	0.999	0.998	0.996	0.992	0.987	0.978	0.963	0.938	0.887	0.761	0.303	***	***	***
96%	1.000	1.000	0.999	0.998	0.997	0.994	0.989	0.982	0.971	0.951	0.910	0.811	0.449	***	***	***
97%	1.000	1.000	1.000	0.999	0.998	0.995	0.992	0.987	0.978	0.963	0.933	0.860	0.591	***	***	***
98%	1.000	1.000	1.000	0.999	0.998	0.997	0.995	0.991	0.986	0.976	0.956	0.907	0.730	***	***	***
99%	1.000	1.000	1.000	1.000	0.999	0.999	0.997	0.996	0.993	0.988	0.978	0.954	0.866	0.073	***	***

Table 2: Ratio of Actual to Targeted Service Level as a Function of the Coefficient of Variation and Targeted Service Level

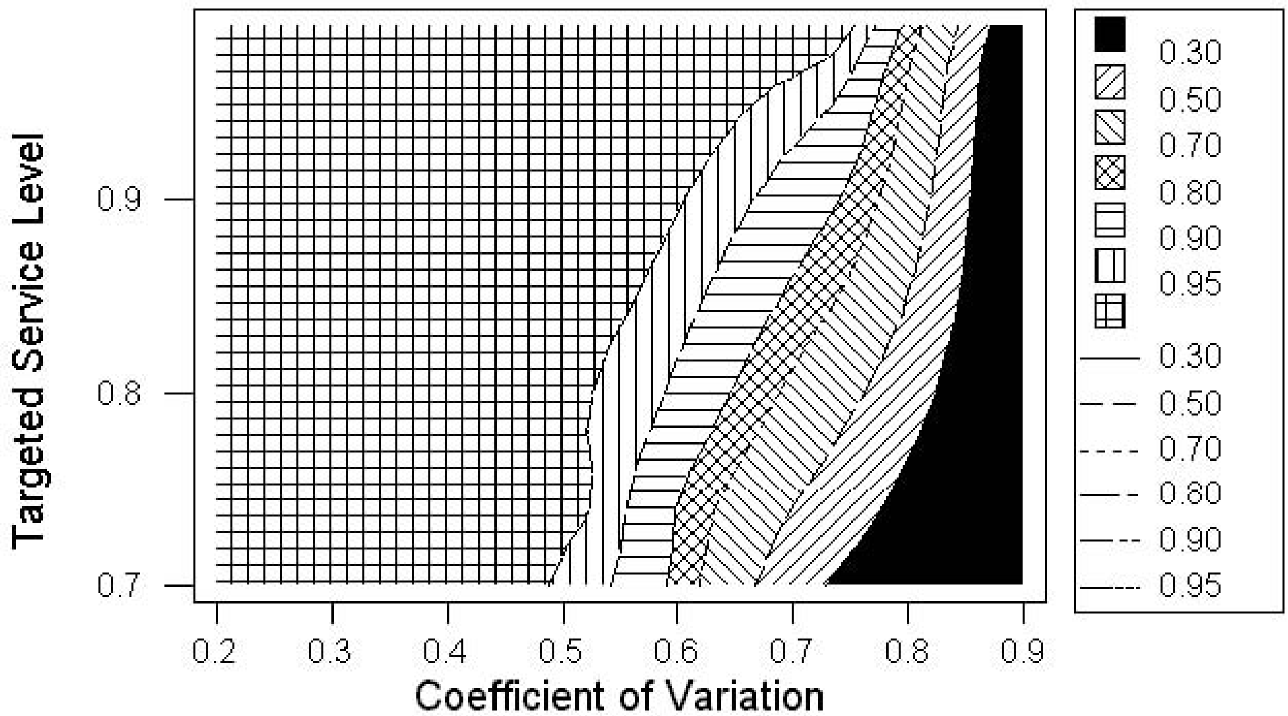


Figure 2: Contour Plot of the Ratio of Actual to Targeted Service Level as a Function of the Targeted Service Level and Coefficient of Variation

Service Level	Ratio of Actual to Targeted Service Level 0.99	0.98	0.97	0.96	0.95	0.90	0.85	0.80	0.75	0.70	0.65	0.60	0.50	0.40	0.30
70%	0.376	0.415	0.442	0.464	0.481	0.542	0.581	0.610	0.632	0.650	0.665	0.678	0.698	0.715	0.728
71%	0.379	0.418	0.446	0.467	0.485	0.547	0.586	0.615	0.637	0.655	0.670	0.682	0.703	0.719	0.732
72%	0.381	0.422	0.449	0.471	0.489	0.551	0.591	0.620	0.642	0.660	0.674	0.687	0.707	0.723	0.736
73%	0.384	0.425	0.453	0.475	0.493	0.556	0.596	0.625	0.647	0.664	0.679	0.692	0.712	0.727	0.740
74%	0.387	0.428	0.457	0.479	0.498	0.561	0.601	0.630	0.652	0.669	0.684	0.696	0.716	0.732	0.744
75%	0.389	0.432	0.461	0.484	0.502	0.566	0.606	0.635	0.657	0.674	0.689	0.701	0.721	0.736	0.748
76%	0.392	0.435	0.465	0.488	0.507	0.571	0.612	0.640	0.662	0.679	0.694	0.706	0.725	0.740	0.752
77%	0.396	0.439	0.469	0.493	0.512	0.577	0.617	0.646	0.667	0.685	0.699	0.711	0.730	0.745	0.756
78%	0.399	0.443	0.474	0.497	0.517	0.583	0.623	0.651	0.673	0.690	0.704	0.716	0.735	0.749	0.761
79%	0.402	0.448	0.479	0.503	0.522	0.588	0.629	0.657	0.679	0.695	0.709	0.721	0.739	0.754	0.765
80%	0.406	0.452	0.484	0.508	0.528	0.595	0.635	0.663	0.684	0.701	0.715	0.726	0.744	0.758	0.769
81%	0.410	0.457	0.489	0.514	0.534	0.601	0.641	0.669	0.690	0.707	0.720	0.731	0.749	0.763	0.774
82%	0.414	0.462	0.494	0.519	0.540	0.607	0.648	0.676	0.696	0.713	0.726	0.737	0.754	0.767	0.778
83%	0.418	0.467	0.500	0.526	0.546	0.614	0.654	0.682	0.703	0.719	0.732	0.742	0.759	0.772	0.783
84%	0.423	0.473	0.507	0.532	0.553	0.622	0.662	0.689	0.709	0.725	0.737	0.748	0.765	0.777	0.787
85%	0.428	0.479	0.513	0.539	0.561	0.629	0.669	0.696	0.716	0.731	0.744	0.754	0.770	0.782	0.792
86%	0.433	0.485	0.520	0.547	0.569	0.637	0.677	0.703	0.723	0.738	0.750	0.760	0.776	0.787	0.797
87%	0.439	0.493	0.528	0.555	0.577	0.646	0.685	0.711	0.730	0.745	0.756	0.766	0.781	0.793	0.802
88%	0.446	0.500	0.537	0.564	0.586	0.655	0.693	0.719	0.737	0.752	0.763	0.772	0.787	0.798	0.807
89%	0.453	0.509	0.546	0.574	0.596	0.664	0.702	0.727	0.745	0.759	0.770	0.779	0.793	0.804	0.812
90%	0.461	0.518	0.556	0.584	0.606	0.674	0.712	0.736	0.753	0.767	0.777	0.786	0.800	0.810	0.818
91%	0.470	0.529	0.567	0.596	0.618	0.685	0.722	0.745	0.762	0.775	0.785	0.793	0.806	0.816	0.824
92%	0.480	0.541	0.580	0.608	0.631	0.697	0.732	0.755	0.771	0.783	0.793	0.801	0.813	0.822	0.830
93%	0.492	0.555	0.594	0.623	0.645	0.710	0.744	0.765	0.781	0.792	0.801	0.809	0.820	0.829	0.836
94%	0.506	0.570	0.611	0.639	0.661	0.724	0.757	0.777	0.791	0.802	0.811	0.818	0.828	0.836	0.843
95%	0.523	0.589	0.630	0.658	0.679	0.740	0.770	0.789	0.803	0.813	0.820	0.827	0.837	0.844	0.850
96%	0.545	0.613	0.653	0.680	0.701	0.758	0.786	0.803	0.815	0.824	0.831	0.837	0.846	0.853	0.858
97%	0.574	0.642	0.681	0.708	0.727	0.779	0.804	0.819	0.830	0.838	0.844	0.849	0.857	0.863	0.868
98%	0.615	0.682	0.719	0.743	0.760	0.804	0.825	0.838	0.847	0.854	0.859	0.863	0.870	0.875	0.879
99%	0.683	0.743	0.773	0.792	0.805	0.839	0.854	0.864	0.870	0.875	0.879	0.883	0.888	0.892	0.896

Table 3: Maximum Allowable Coefficient of Variation as a Function of Acceptable Ratio of Actual to Targeted Service Level and Service Level

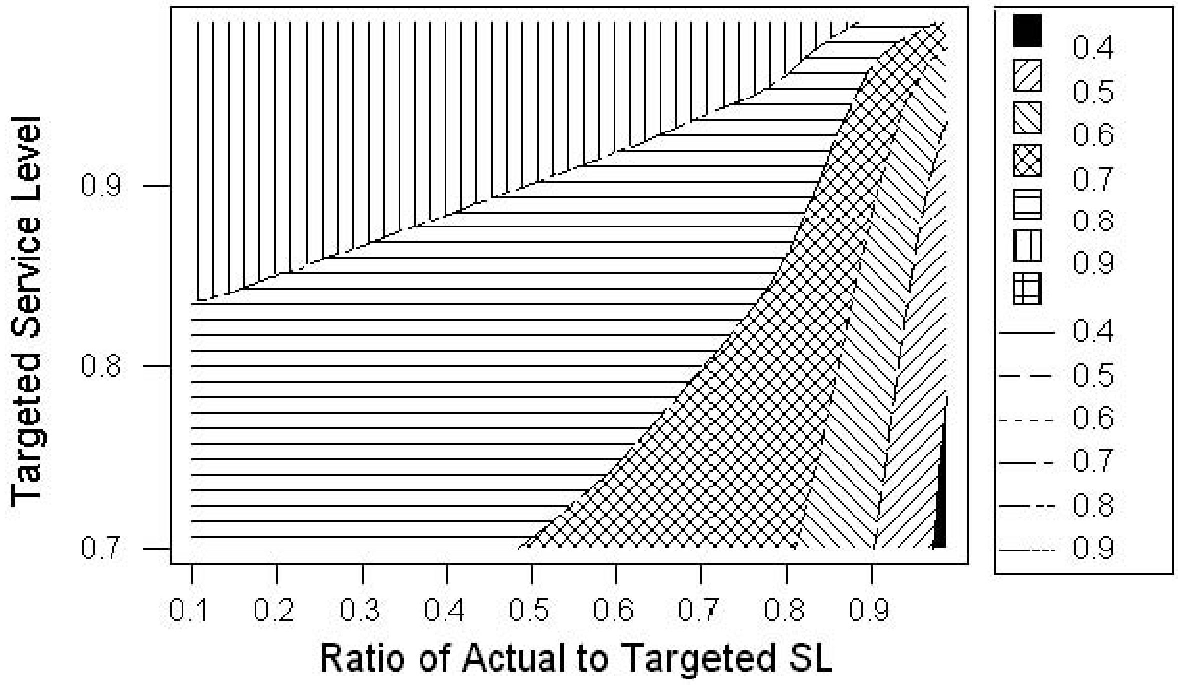


Figure 3: Contour Plot of the Maximum Allowable ("Critical") Coefficient of Variation as a Function of the Targeted Service Level and Acceptable Ratio of Actual to Targeted Performance

CONCLUSION

This paper has presented an improved methodology for establishing safety stocks through the use of the standardized, left-truncated normal distribution as the assumed demand probability distribution. Tables have been provided which facilitate the use of this methodology and illustrate the degradation in achieved service level that occurs when conventional (non-truncated normal distribution) methods are used for demand patterns that reach "critical" values of the coefficient of variation (indicative of a high degree of truncation). Through the use of this methodology, safety levels that are more reflective of the actual demand patterns can be established and targeted service levels can be achieved.

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