The Queueing Theory of the Erlang Distributed Interarrival and Service Time

Noln Plumchitchom
Nick T. Thomopoulos
Stuart School of Business, Illinois Institute of Technology
Email: noln@stuart.iit.edu, thomop@stuart.iit.edu

ABSTRACT

This paper is concerned with the study of non-Markovian queueing systems, contributed mainly on Erlang distributed interarrival and service time that expanded the knowledge beyond Poisson arrival and departure rate. We introduce a new method of *Possible Probability Stages Searching Algorithm* (PSA) and apply stochastic algebra, calculus, and the theory of Martingales to derive system measures that before never have been obtained. We also utilize the method of simulation to further estimate the point beyond today's computational ability.

INTRODUCTION

The Erlang distribution can be used to model interarrival and service time with a low coefficient of variation (0.0 < *Cov* < 1.0). The principle objective of this paper is to develop tables of the standardized queueing statistics to facilitate the arbitrary interarrival and service time. In order to yield more precise system measures for any statistic distribution that differs from exponential, for use and reference by workers/researchers. And also introduce a new method of PSA invented for this situation.

This paper is partially based on the Ph.D. dissertation and research, *The Queueing Theory of the Erlang Distributed Interarrival and Service Time* [1]. The dissertation shows how to calculate the entire queueing statistic determined by *Cov* that range from 1.00 to 0.32, and simulation technique to model *Cov* further to 0.00. This research has evolved from the results of Thomopoulos [2] where he presents a wide variety of queueing systems in four chapters and lists the queueing statistics of these systems.

The methodology used in this paper was based on previous research conducted by Thomopoulos, McManamon and Janc [3] and by Janc, Thomopoulos and Marks [4]. The research was for the U.S. Navy concerning various voice radio networks and alternatives to the present supervisory control techniques. Various queueing systems were formulated and led to a large and complex set of equations. The associated Markoff process was specified by various states and a state transition probability matrix. Often the matrices became very large and unsolvable by the then techniques. This research developed a technique to obtain the numerical results that were needed. The technique separated the matrix structure into a series of sub-matrices that could be solved one at a time. This method allowed the research to tackle very complex Naval networks and allowed extending the inter-arrival and service times to distributions beyond the Exponential and on its way to the Erlang. This research was given the *29th Military Operations Research Society Rist Prize Award Citation* for the year 1972.

The Requirement of the Occupation Rate (ρ)

Consider a single-server queue of Erlang distributed interarrival and service times with mean ($1/\lambda$) and ($1/\mu$) respectively. The unit has to pass, stage by stage, through a series of K_a and K_d independent arrival and service stages, where each stage takes an exponentially distributed time, each unit is served in order of arrival (Kendall's notation: $E_{Ka}/E_{Kd}/1$). For stability we required that ρ, the fraction of time the service facility is working

$$\rho = \lambda/\mu \quad (1)$$

is less than one.

The Description of the Probability Stage (P_{nij})

To describe the state of a non empty system is by the trio (*n, i, j*) where *n* denotes the number of units in the system, *i* the arrival phase of the unit upon arrival stage and *j* the service phase of the unit in service stage, denoted by P_{nij}.

Equilibrium Distribution Probabilities

For the one-dimensional phase description for $E_2/E_2/1/2$ model with maximum number allowance in the system is 2, we get the flow diagram of Figure 1, the general diagram of $E_{Ka}/E_{Kd}/1$ can not be drawn unless the value of K_a and K_d are known and the PSA has been employed to acquire all the possible transitions between all probability stages (P_{nij}s), the calculation of $E_{Ka}/E_{Kd}/1$ will be shown in a later section.

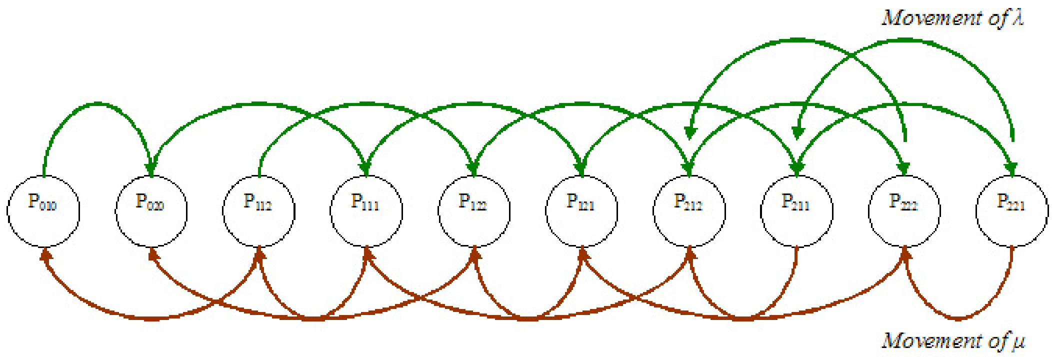

Figure 1: One-dimensional flow diagram for the $E_2/E_2/1/2$ model

The transition from phase to phase within each stage of arrival and departure have been governed by the exponentially distributed mean interarrival ($1/\lambda_{Exponential}$) and departure ($1/\mu_{Exponential}$) time which are equal to $1/K_a\lambda_{Erlang}$ and $1/K_d\mu_{Erlang}$ of the Erlang distribution respectively [1]. From the global balance principal we can equate the flow out of, and into stage P_{nij} to obtain the following set of equilibrium equations for P_n. For the $E_2/E_2/1/2$ model, we have the following:

$$\left.\begin{array}{ll}
P_{010}: & 0 = -K_a\lambda P_{010} + K_d\mu P_{112} \\
P_{020}: & 0 = -K_a\lambda P_{020} + K_a\lambda P_{010} + K_d\mu P_{122} \\
P_{111}: & 0 = (-K_a\lambda - K_d\mu) P_{111} + K_a\lambda P_{020} + K_d\mu P_{212} \\
P_{112}: & 0 = (-K_a\lambda - K_d\mu) P_{112} + K_d\mu P_{111} \\
P_{121}: & 0 = (-K_a\lambda - K_d\mu) P_{121} + K_a\lambda P_{111} + K_d\mu P_{222} \\
P_{122}: & 0 = (-K_a\lambda - K_d\mu) P_{122} + K_a\lambda P_{112} + K_d\mu P_{121} \\
P_{211}: & 0 = (-K_a\lambda - K_d\mu) P_{211} + K_a\lambda P_{121} + K_a\lambda P_{221} \\
P_{212}: & 0 = (-K_a\lambda - K_d\mu) P_{212} + K_a\lambda P_{122} + K_d\mu P_{211} + K_a\lambda P_{222} \\
P_{221}: & 0 = (-K_a\lambda - K_d\mu) P_{221} + K_a\lambda P_{211} \\
P_{222}: & 0 = (-K_a\lambda - K_d\mu) P_{222} + K_a\lambda P_{212} + K_d\mu P_{221}
\end{array}\right\} \quad (2)$$

rewrite the Equation 2 with the value of K_a and K_d = 2 for $E_2/E_2/1/2$ model into the following matrix relation, AP = BP_{010} (with one redundant equation omitted):

$$\begin{bmatrix} 0 & 0 & +2\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\lambda & 0 & 0 & 0 & +2\mu & 0 & 0 & 0 & 0 \\ +2\lambda & -2(\lambda+\mu) & 0 & 0 & 0 & 0 & +2\mu & 0 & 0 \\ 0 & +2\mu & -2(\lambda+\mu) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +2\lambda & 0 & -2(\lambda+\mu) & 0 & 0 & 0 & 0 & +2\mu \\ 0 & 0 & +2\lambda & +2\mu & -2(\lambda+\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +2\lambda & 0 & -2(\lambda+\mu) & 0 & +2\lambda & 0 \\ 0 & 0 & 0 & 0 & +2\lambda & +2\mu & -2(\lambda+\mu) & 0 & +2\lambda \\ 0 & 0 & 0 & 0 & 0 & +2\lambda & 0 & -2(\lambda+\mu) & 0 \end{bmatrix} \begin{bmatrix} P_{020} \\ P_{111} \\ P_{112} \\ P_{121} \\ P_{122} \\ P_{211} \\ P_{212} \\ P_{221} \\ P_{222} \end{bmatrix} = \begin{bmatrix} 2\lambda \\ -2\lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} P_{010}$$

then P = $(A^{-1}B)P_{010}$ = QP_{010}:

$$\begin{bmatrix} P_{020} \\ P_{111} \\ P_{112} \\ P_{121} \\ P_{122} \\ P_{211} \\ P_{212} \\ P_{221} \\ P_{222} \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 & +2\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\lambda & 0 & 0 & 0 & +2\mu & 0 & 0 & 0 & 0 \\ +2\lambda & -2(\lambda+\mu) & 0 & 0 & 0 & 0 & +2\mu & 0 & 0 \\ 0 & +2\mu & -2(\lambda+\mu) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +2\lambda & 0 & -2(\lambda+\mu) & 0 & 0 & 0 & 0 & +2\mu \\ 0 & 0 & +2\lambda & +2\mu & -2(\lambda+\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +2\lambda & 0 & -2(\lambda+\mu) & 0 & +2\lambda & 0 \\ 0 & 0 & 0 & 0 & +2\lambda & +2\mu & -2(\lambda+\mu) & 0 & +2\lambda \\ 0 & 0 & 0 & 0 & 0 & +2\lambda & 0 & -2(\lambda+\mu) & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2\lambda \\ -2\lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) P_{010} = \begin{bmatrix} Q_{020} \\ Q_{111} \\ Q_{112} \\ Q_{121} \\ Q_{122} \\ Q_{211} \\ Q_{212} \\ Q_{221} \\ Q_{222} \end{bmatrix} P_{010}$$

calculate all values of probability of n units in the system (P_n), for n = 0 to M (in this situation, M = 2):

from $\Sigma P_n = 1$, for n = 0 to M, we have

$P_{010} = 1/(1 + Q_{020} + Q_{112} + Q_{111} + Q_{122} + Q_{121} + Q_{212} + Q_{211} + Q_{222} + Q_{221})$
$P_{020} = Q_{020} P_{010}$, $P_{112} = Q_{112} P_{010}$, $P_{111} = Q_{111} P_{010}$, $P_{122} = Q_{122} P_{010}$, $P_{121} = Q_{121} P_{010}$,
$P_{212} = Q_{212} P_{010}$, $P_{211} = Q_{211} P_{010}$, $P_{222} = Q_{222} P_{010}$, $P_{221} = Q_{221} P_{010}$
$P_0 = P_{010} + P_{020}$, $P_1 = P_{112} + P_{111} + P_{122} + P_{121}$, $P_2 = P_{212} + P_{211} + P_{222} + P_{221}$ (3)

calculate the effective rate of arrival (λ_e):

$$\lambda_e = \lambda (1-P_M) \quad (4)$$

calculate system effectiveness (K_e):

$$K_e = \lambda_e / \lambda \quad (5)$$

to replicate the $E_2/E_2/1/\infty$ or $E_2/E_2/1$ model, we need to find $E_2/E_2/1/M$ model that has:

$$P_M \approx P_{M+1}, P_{M+2}, \ldots, P_\infty \approx 0$$

by adding the maximum allowed in the system (M) by 1 and recalculate the system effectiveness (K_e) until the value of K_e approachs 1. From our previous illustration above we found $E_2/E_2/1/12$ is more than 99.9995% certain that this model is no difference to $E_2/E_2/1/\infty$ or $E_2/E_2/1$ model with value of P_{12} approaches $P_\infty = 0$, yield the results of $P_0 = 0.40$, $P_1 = 0.36$, $P_2 = 0.15$, $P_3 = 0.06$, … , $P_{12} = 0.00$, $\lambda_e = 0.6$ for Erlang distributed mean interarrival time of 1.6667.

calculate coefficient of variation (*Cov*)

$$\left.\begin{aligned} Cov_{ta} &= 1/(K_a)^{1/2} = 0.71 \\ Cov_{ts} &= 1/(K_d)^{1/2} = 0.71 \end{aligned}\right\} \quad (6)$$

Mean System Performance Measures

With the probabilities associated with the system having been obtained, it is then possible to measure the average number of units in the queue, the service facility and the total system. These units are commonly denoted in queueing theory by L_q, L_s, and L, respectively, and are calculated as:

$$L_q = \Sigma(n\text{-}1)P_n \quad \text{for } n \geq 1 \quad (7)$$
$$L_s = \rho \quad (8)$$
$$L = L_q + L_s \quad (9)$$

Another common set of measures that are useful in queueing theory are measures of the average time a unit is in the queue, the service facility of the system in total. These measures are commonly labeled as W_q, W_s, and W, respectively, and are obtained in the following manner:

$$W_q = L_q/\lambda_e \quad (10)$$
$$W_s = L_s/\lambda_e \quad (11)$$
$$W = L/\lambda_e \quad (12)$$

note also that $W_s = 1/\mu$ and that $W = W_q + W_s$

Recall from the above that W_q represents the average amount of time a unit is in the queue. This average is based on all units whether or not they are delayed and wait in the queue in the first place. Any unit not waiting on the queue will have zero as its queue time. For those units that are delayed in the queue, the average time is designated as W_q'. This latter measure is determined by the following relation:

$$W_q' = W_q/(1\text{-} P_0) \quad (13)$$

A final measure that may be of the interest is here labeled as the service level *SL*. In this situation, the service level represents the percent of arriving units that find a service facility open and need not be delayed in the queue. This is obtained by the relation:

$$SL = P_0 \quad (14)$$

Understanding the Transition Driven by Lamda (λ) and Mu (μ)

To better construct $E_{Ka}/E_{Kd}/1$ model, we found that it is easier to derive the equilibrium equation from the flow diagram shown previously. The following sections will show the transition in each phase of arrival and departure that been driven by λ and μ within the logical constrains.

Transition that Governed by Lamda (λ) in the Arrival Stage to Queue Line to Departure Stage

The unit in the arrival stage that has not reached the final phase of arrival is allowed to move to next phase of arrival.

The unit that reaches the final phase of arrival when there is no other unit in any phase of departure is allowed to move to the first phase of departure without being queued, creating additional number of unit in the system. With the vacancy in any phase of arrival, the next new unit is allowed to enter the first phase of arrival.

The unit that reaches the final phase of arrival when there is another unit in any phase of departure and when the number allowance in the system has reached maximum or full queue line, the unit has to leave the system with out going through the departure stage so that the new unit is allowed to enter the first phase of arrival.

The unit that reaches the final phase of arrival and there is another unit in any phase of departure with the number allowance in the system has not reached maximum or the unit can be put into queue, the unit is allowed to enter into the queueing process, that leave the empty stage of arrival therefore the next new unit can now be entered into the first phase of arrival.

Transition that Governed by Mu (μ) in the Departure Stage to Stage of Leaving the System

The unit in the departure stage that has not reached the final phase of departure is allowed to move to next phase of departure.

The unit that reaches the final phase of departure and it is the only unit in the system, means that there is no queue waiting to enter the first phase of departure then the unit is allowed to leave the system with empty stage of departure.

The unit that reaches the final phase of departure when more than one unit is in the system, imply that there is queue waiting to enter the first phase of departure, the unit is allowed to leave the system which present empty stage of departure therefore allowing the first in queue unit to enter the first phase of departure.

There is no movement is the stage of departure if there is only a unit in the arrival stage and none in the system.

POSSIBLE PROBABILITY STAGES SEARCHING ALGORITHM (PSA)

From the previous section we now understand the allowable movement of the unit in the system that is governed by λ and μ, we can now construct PSA based on the above descriptive reasoning to further help us accurately draw the flow diagram of the transition in order to derive the equilibrium equation. Starts with P_{010} as the beginning, the algorithms are as follow:

Step 1: Transition of the unit in each stage that governed by λ

If $i < K_a$ Then $i = i + 1$ (Step 1.1)
Else If $i = K_a$ And $j = 0$ Then $i = 1, j = 1$ and $n = n + 1$ (Step 1.2)
Else If $i = K_a$ And $j > 0$ And $n = M$ Then $i = 1$ (Step 1.3)
Else If $i = K_a$ And $j > 0$ And $n \neq$ M Then $i = 1$ and $n = n + 1$ (Step 1.4)

Step 2: Store result of λ transition between P_{nij}s

Step 3: If the transition found to be the existing probability stage (P_{nij}), this means that the probability stage has transited back to existing probability stage by λ, and therefore we can proceed to the next step. Otherwise proceed back to Step 1.

Step 4: Transition of the unit in each stage that governed by μ, from all P_{nij}s that transited by λ

If $j < K_d$ Then $j = j + 1$ (Step 4.1)
Else If $j = K_d$ And $n = 1$ Then $j = 0$ and $n = n – 1$ (or $n = 0$) (Step 4.2)
Else If $j = K_d$ And $n > 1$ Then $j = 1$ and $n = n – 1$ (Step 4.3)
Else If $n = 0$ And $i > 0$ And $j = 0$ (no transition) (Step 4.4)

Step 5: Store result of the transition that governed by μ between P_{nij}s

Step 6: If the transition found to be the existing probability stage (P_{nij}), this means that the probability stage has transited back to existing probability stage by μ, and therefore we can proceed to the next step, if and only if all the P_{nij}s have been transited by μ. Otherwise proceed back to Step 4.

Step 7: Find all value of P_{nij}s created but have not been transited by λ

Step 8: Transition of the unit in each stage that governed by λ from created P_{nij}s by μ in Step 7, do until all value of created P_{nij}s have been transited

If $i < K_a$ Then $i = i + 1$ (Step 8.1)
Else If $i = K_a$ And $j > 0$ And $n =$ M Then $i = 1$ (Step 8.2)
Else If $i = K_a$ And $j > 0$ And $n \neq$ M Then $i = 1$ and $n = n + 1$ (Step 8.3)

Step 9: Store result of λ transition between P_{nij}s

Step 10: If the transition found to be the existing probability stage (P_{nij}), this means that the probability stage has transited back to existing probability stage by λ, and therefore we have completed the probability stages search algorithm, if and only if all the P_{nij}s have been transited by λ. Otherwise proceed back to Step 8.

FLOW DIAGRAM

From the PSA, Step 1 to Step 3, we found the possible probability stages (P_{nij}s) for $E_2/E_2/1/2$ as follow (the larger model is too complex to show in this paper):

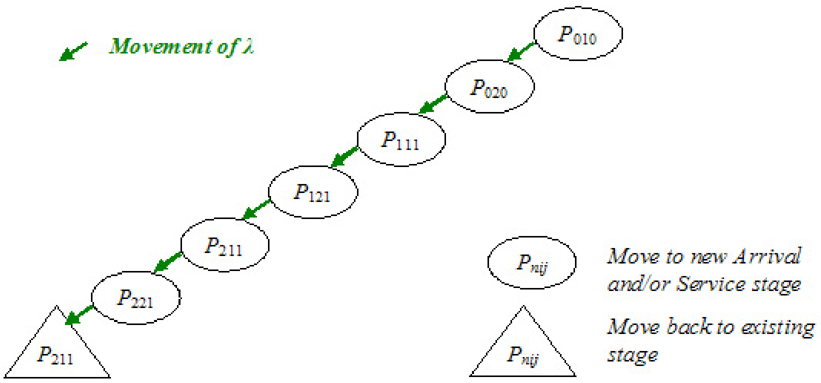

Figure 2: One-dimensional flow diagram for the $E_2/E_2/1/2$ from searching algorithm Step 1 to Step 3

the searching algorithm, Step 4 to Step 6 yield the following:

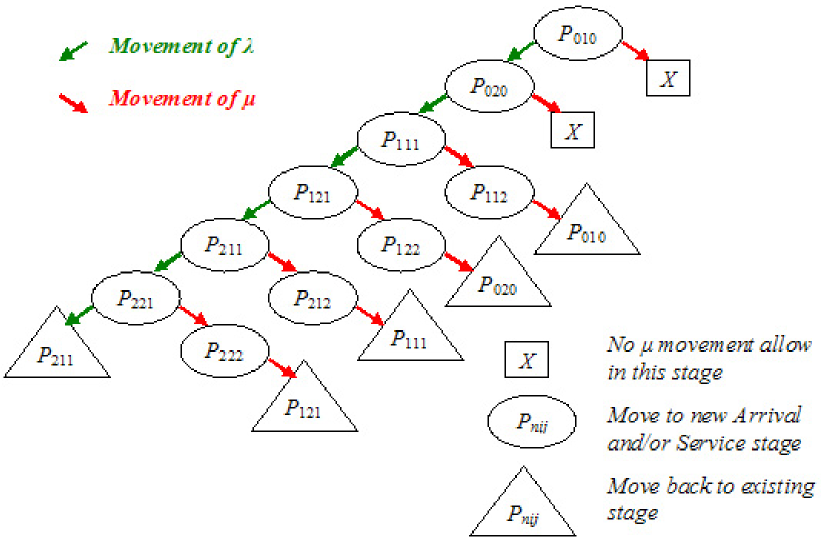

Figure 3: One-dimensional flow diagram for the $E_2/E_2/1/2$ from searching algorithm Step 4 to Step 6

we have completed the search algorithm with Step 7 to Step 10, and have all the possible probability stages for $E_2/E_2/1/2$ model as follow:

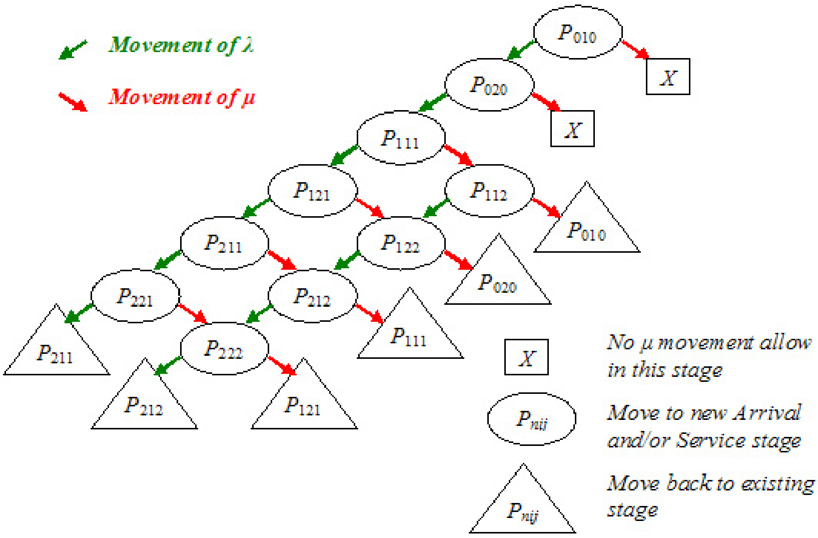

Figure 4: One-dimensional flow diagram for the $E_2/E_2/1/2$ from searching algorithm Step 7 to Step 10

Summary of System Measures Tables

From all possible probability stages (Figure 2 to Figure 4 copied the logic), we can construct the alternate completed flow diagram (Figure 1) to assist us in determine the equilibrium equations (Equation 2) to acquire value of K_e (Equation 3 to Equation 5), and adding M, recalculate (from beginning), until value $0.999995 < K_e < 1.00$ satisfied, in order to derive all the system measures for $E_{Ka}/E_{Kd}/1/\infty$. Table 1 to Table 6 give statistics on P_0, L_q, L, W, W_q , and W_q', respectively.

Table 1: The Probability of No Unit in the System (P_0) for One Service Facility with Arbitrary Interarrival and Service Times

P_0		ρ								
Cov_{Ta}	Cov_{Ts}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0	0.0	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10
0.0	0.5	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10
0.0	1.0	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10
0.5	0.0	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10
0.5	0.5	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10
0.5	1.0	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10
1.0	0.0	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10
1.0	0.5	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10
1.0	1.0	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10

Table 2: Average Number of Units in the Queue (L_q) for One Service Facility with Arbitrary Interarrival and Service Times

L_q		ρ								
Cov_{Ta}	Cov_{Ts}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0	0.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.0	0.5	0.00	0.00	0.00	0.00	0.01	0.03	0.09	0.26	0.83
0.0	1.0	0.00	0.00	0.01	0.05	0.13	0.29	0.61	1.35	3.76
0.5	0.0	0.00	0.00	0.00	0.01	0.02	0.06	0.13	0.31	0.90
0.5	0.5	0.00	0.00	0.01	0.03	0.06	0.14	0.30	0.67	1.87
0.5	1.0	0.00	0.01	0.04	0.10	0.22	0.44	0.87	1.81	4.84
1.0	0.0	0.01	0.02	0.06	0.13	0.25	0.45	0.82	1.60	4.00
1.0	0.5	0.01	0.03	0.08	0.17	0.31	0.56	1.02	2.00	5.06
1.0	1.0	0.01	0.05	0.13	0.27	0.50	0.90	1.63	3.20	8.10

Table 3: Average Number of Units in the System (L) for One Service Facility with Arbitrary Interarrival and Service Times

L		ρ								
Cov_{Ta}	Cov_{Ts}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0	0.0	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.0	0.5	0.10	0.20	0.30	0.40	0.51	0.63	0.79	1.06	1.73
0.0	1.0	0.10	0.20	0.31	0.45	0.63	0.89	1.32	2.15	4.66
0.5	0.0	0.10	0.20	0.30	0.41	0.52	0.66	0.83	1.11	1.80
0.5	0.5	0.10	0.20	0.31	0.43	0.56	0.74	1.00	1.47	2.77
0.5	1.0	0.10	0.21	0.34	0.50	0.72	1.04	1.57	2.61	5.74
1.0	0.0	0.11	0.22	0.36	0.53	0.75	1.05	1.52	2.40	4.90
1.0	0.5	0.11	0.23	0.38	0.57	0.81	1.16	1.72	2.80	5.96
1.0	1.0	0.11	0.25	0.43	0.67	1.00	1.50	2.33	4.00	9.00

Table 4: Standardized Average Time a Unit is in the System (W) for One Service Facility with Arbitrary Interarrival and Service Times

W		ρ								
Cov_{Ta}	Cov_{Ts}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0	0.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.0	0.5	1.00	1.00	1.00	1.00	1.02	1.05	1.13	1.32	1.92
0.0	1.0	1.00	1.01	1.04	1.12	1.25	1.48	1.88	2.69	5.18
0.5	0.0	1.00	1.00	1.01	1.02	1.05	1.10	1.19	1.38	2.00
0.5	0.5	1.00	1.01	1.03	1.07	1.13	1.24	1.43	1.84	3.08
0.5	1.0	1.01	1.04	1.12	1.24	1.43	1.73	2.24	3.27	6.38
1.0	0.0	1.06	1.12	1.21	1.33	1.50	1.75	2.16	3.01	5.45
1.0	0.5	1.07	1.16	1.27	1.42	1.62	1.94	2.46	3.50	6.62
1.0	1.0	1.11	1.25	1.43	1.67	2.00	2.50	3.33	5.00	10.00

Table 5: Standardized Average Time a Unit is in the Queue (W_q) for One Service Facility with Arbitrary Interarrival and Service Times

W_q		ρ								
Cov_{Ta}	Cov_{Ts}	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
0.0	**0.0**	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.0	**0.5**	0.00	0.00	0.00	0.00	0.02	0.05	0.13	0.32	0.92
0.0	**1.0**	0.00	0.01	0.04	0.12	0.25	0.48	0.88	1.69	4.17
0.5	**0.0**	0.00	0.00	0.01	0.02	0.05	0.10	0.19	0.38	1.00
0.5	**0.5**	0.00	0.01	0.03	0.07	0.13	0.24	0.43	0.84	2.08
0.5	**1.0**	0.01	0.04	0.12	0.24	0.43	0.73	1.24	2.27	5.38
1.0	**0.0**	0.06	0.12	0.21	0.33	0.50	0.75	1.16	2.01	4.45
1.0	**0.5**	0.07	0.16	0.27	0.42	0.62	0.94	1.46	2.50	5.62
1.0	**1.0**	0.11	0.25	0.43	0.67	1.00	1.50	2.33	4.00	9.00

Table 6: Standardized Average Time a Delayed Unit is in the Queue (W_q') for One Service Facility with Arbitrary Interarrival and Service Times

W_q'		ρ								
Cov_{Ta}	Cov_{Ts}	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
0.0	**0.0**	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.0	**0.5**	0.00	0.00	0.00	0.01	0.03	0.09	0.19	0.40	1.03
0.0	**1.0**	0.00	0.04	0.14	0.30	0.51	0.80	1.25	2.11	4.64
0.5	**0.0**	0.00	0.01	0.03	0.06	0.10	0.16	0.27	0.48	1.11
0.5	**0.5**	0.01	0.04	0.09	0.16	0.26	0.40	0.62	1.05	2.31
0.5	**1.0**	0.07	0.22	0.40	0.61	0.87	1.21	1.77	2.83	5.98
1.0	**0.0**	0.56	0.62	0.71	0.83	1.00	1.25	1.66	2.51	4.95
1.0	**0.5**	0.69	0.78	0.89	1.04	1.25	1.56	2.08	3.12	6.25
1.0	**1.0**	1.11	1.25	1.43	1.67	2.00	2.50	3.33	5.00	9.99

the non displayed system statistics, L_s, W_s, and SL, is because there are no significant change in measures value (See Equation 8, Equation 11, and Equation 14). The completed results of all system measures derived from $E_1/E_1/1$ ($Cov_{ta} = Cov_{ts} = 1.0$) to $E_\infty/E_\infty/1$ ($Cov_{ta} = Cov_{ts} = 0.0$), can be found at *The Queueing Theory of the Erlang Distributed Interarrival and Service Time*. Chicago, IL: Illinois Institute of Technology Stuart School of Business, 2006 [1]. Thomopoulos [2] gives similar table of $Cov_{ta} = 1$ and Cov_{ts} range from 0.0 to 2.0.

After we derived all of the system measures from $E_1/E_1/1$ to $E_{10}/E_{10}/1$ ($Cov = 1.00$ to $Cov = 0.32$), we explored the system capacity (M, regarding to the interarrival and service time distribution) and number of variables or number of equilibrium equations (nEq) needed. This was accomplished by using regression analysis, whereby we derived the estimated M and nEq as follow:

$$\hat{M} = 67.78 - 3.40K_a - 3.41K_d \quad (15)$$

$$n\hat{E}q = 63.82K_a + 63.17K_d - 0.28M - 77.14 \quad (16)$$

from Equation 16 together with Equation 15 and Equation 6 we can say that:

$$n\hat{E}q = 64.77K_a + 64.12K_d - 36.12 \quad (17)$$

$$= 64.77(Cov_{Ta}^{-2}) + 64.12(Cov_{Ts}^{-2}) - 36.12 \quad (18)$$

These equations are assisting us in concluding that even though the corresponding equilibrium equation is linear in term of K_a and K_d, but since we are interested in deriving Cov tables, the estimated nEq does exhibit the increase rate of exponential (given $0.0 < Cov < 1.00$).

This shows us that, for *Cov* of: (0.30, 0.25, 0.20, 0.15, 0.10, and 0.05) the associated K of (11, 16, 25, 44, 100, and 400), respectively. And thereby the corresponding numbers of variables are 1,396, 2,026, 3,186, 5,692, 12,853, and 51,520 which will lead to number of matrix components of 1,948,791, 4,105,162, 10,151,424, 32,402,558, 165,196,524, and 2,654,298,035, respectively. Note this approaches infinity as *Cov* approaches zero, therefore from the point beyond $E_{10}/E_{10}/1$ we preferred the method of simulation. Figure 5 and Figure 6 demonstrate that even though the estimated number of equations and matrix components from K_a and K_d are linear, the one from *Cov* is increasing in exponential like function.

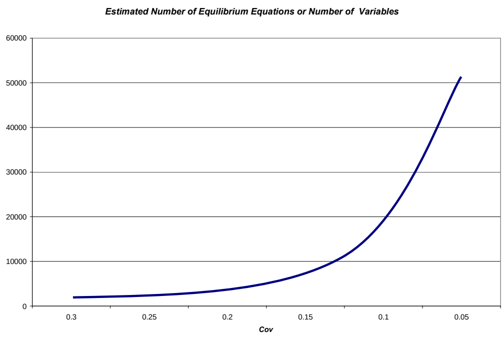

Figure 5: Estimated Number of Equilibrium Equations

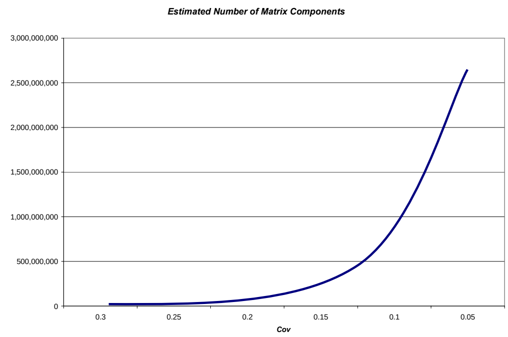

Figure 6: Estimated Number of Matrix Components Use in Calculation

SIMULATION

A unit in Erlang distributed interarrival and service time queueing system has to pass, phase by phase, through a series of independent exponentially distributed arrival and departure phases, K_a and K_d respectively. Therefore we can model $E_{Ka}/E_{Kd}/1$ queue similar to that of the $M/M/1$ queue with arrival and service rate of the following:

$$\lambda_{Exponential} = \lambda_{Erlang} \text{ x } K_a \qquad (19)$$

$$\mu_{Exponential} = \mu_{Erlang} \text{ x } K_d \qquad (20)$$

we can further simulate the interarrival time (Ta) and service time (Ts) in each independent stage (i) by:

$$Ta_{Exponential}(i) = [\text{Log}(1- Rnd(i))]/(-\lambda_{Exponential}) \quad \text{for } i = 1 \text{ to } K_a \qquad (21)$$

$$Ts_{Exponential}(i) = [\text{Log}(1- Rnd(i))]/(-\mu_{Exponential}) \quad \text{for } i = 1 \text{ to } K_d \qquad (22)$$

then combining each independent stage (i) to form up an Erlang interarrival time ($Ta_{Erlang}(j)$) and service time ($Ts_{Erlang}(j)$) as follow:

$$Ta_{Erlang}(j) = \Sigma Ta_{Exponential}(i) \quad \text{for } i = 1 \text{ to } K_a\text{, do until } j = 1 \text{ to } l \qquad (23)$$

$$Ts_{Erlang}(j) = \Sigma Ts_{Exponential}(i) \quad \text{for } i = 1 \text{ to } K_d\text{, do until } j = 1 \text{ to } l \qquad (24)$$

for the constant distribution, we have:

$$Ta_{Erlang}(j) = 1/\lambda_{Erlang} \quad \text{do until } j = 1 \text{ to } l \qquad (25)$$

$$Ts_{Erlang}(j) = 1/\mu_{Erlang} \quad \text{do until } j = 1 \text{ to } l \qquad (26)$$

time of the unit (j) enters the system ($Ti(j)$) is calculated as:

$$Ti(j) = Ti(j - 1) + Ta_{Erlang}(j)\text{, given } Ti(0) = 0\text{, do until } j = 1 \text{ to } l \qquad (27)$$

time of the unit (j) begins service ($Tb(j)$) is equal to:

$$Tb(j) = Ti(j) \quad \text{do until } j = 1 \text{ to } l \tag{28.1}$$

If the next unit (j) arrived before the previous unit ($j – 1$) leaves the service facility ($Tb(j) < Te(j - 1)$), the unit (j) must wait until the previous unit ($j – 1$) finished, then the time of the unit (j) begins service is obtained in the following manner:

$$Tb(j) = Te(j - 1), \text{given} \quad Te(0) = 0, \text{ do until } j = 1 \text{ to } l \tag{28.2}$$

time of the unit (j) leaves the service facility or time of service end ($Te(j)$) is determined by the following relation:

$$Te(j) = Tb(j) + Ts_{Erlang}(j) \tag{29}$$

the summation of all interarrival time (TTa_{Erlang}) and service time (TTs_{Erlang}) are obtained by the relation:

$$TTa_{Erlang} = \Sigma\, Ta_{Erlang}(j) \quad \text{for } j = 1 \text{ to } l \text{ or } Ti(l) \tag{30}$$

$$TTs_{Erlang} = \Sigma\, Ts_{Erlang}(j) \quad \text{for } j = 1 \text{ to } l \tag{31}$$

the average of all interarrival time (ATa_{Erlang}) and the service time (ATs_{Erlang}) are calculated as:

$$ATa_{Erlang} = [\Sigma\, Ta_{Erlang}(j)]/l \quad \text{for } j = 1 \text{ to } l \text{ or } Ti(l)/l \tag{32}$$

$$ATs_{Erlang} = [\Sigma\, Ts_{Erlang}(j)]/l \quad \text{for } j = 1 \text{ to } l \tag{33}$$

the total time the system idle (*Tidle*) is the summation of the difference between time of unit (j) arrives in the system and time of previous unit ($j – 1$) depart the system, for all l units:

$$Tidle = \Sigma[Tb(j) – Te(j – 1)] \text{ or } \Sigma Tb(j) – \Sigma Te(j – 1) \quad \text{for } j = 1 \text{ to } l \tag{34}$$

the total time of each single unit waits in queue (Tq) is the summation of the difference between time of the unit (j) allows to begin service ($Tb(j)$) and time of the unit (j) enters system ($Ti(j)$):

$$Tq = \Sigma[Tb(j) – Ti(j)] \text{ or } \Sigma Tb(j) – \Sigma Ti(j) \quad \text{for } j = 1 \text{ to } l \tag{35}$$

Mean System Performance Measures from Simulation

With the simulated actual system arrival and departure, it is then possible to measure the probability of no unit in the system (P_0) as follow:

$$P_0 = Tidle/Te(l) \tag{36}$$

Another common set of measures, the average number of units in the queue (L_q), the service facility (L_s) and the total system (L), are calculated as:

$$L_q = Tq/Te(l) \tag{37}$$

$$L_s = TTs_{Erlang}/Te(l) \tag{38}$$

$$L = L_q + L_s \tag{39}$$

For measuring the average of time a unit is in the queue (W_q), the service facility (W_s) or the system in total (W). These measures are obtained in the following manner:

$$W_q = Tq/l \tag{40}$$

$$W_s = ATs_{Erlang} \tag{41}$$

$$W = W_q + W_s \tag{42}$$

For the average time of those units that are delayed in the queue (W_q'), is determined by the following relation:

$$W_q' = W_q/(1 - P_0) \qquad (43)$$

A final measure, service level (SL), the percent of arriving units that find a service facility open and need not be delayed in the queue. This is obtained by the relation:

$$SL = P_0 \qquad (44)$$

ANALYSIS

From the simulated system measures with large sample (l = 10 million), we found that the results (99% accuracy) are not significantly different from that of the PSA (99.9995% accuracy), mentioned earlier for $E_1/E_1/1$ ($M/M/1$, Cov = 1.00) to $E_{10}/E_{10}/1$ (Cov = 0.32). Therefore we further estimate $E_{10+}/E_{10+}/1$ (Cov < 0.32) and yield the queueing statistics that are summarized in Table 1 to Table 6.

The Non Decreasing Function of *M*

After we calculate the Erlang distributed interarrival and service time queueing system with PSA, we found that the value of M, has exhibited the non decreases function shown in the following figures:

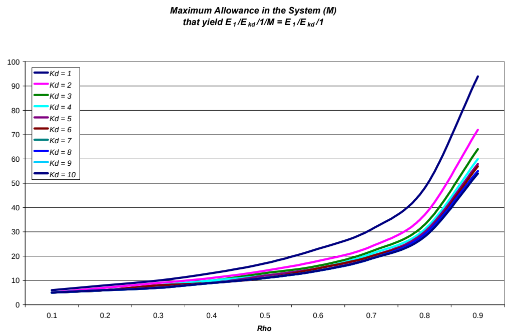

Figure 7: The non decreasing function of M for given K_a of 1 and K_d from 1 to 10

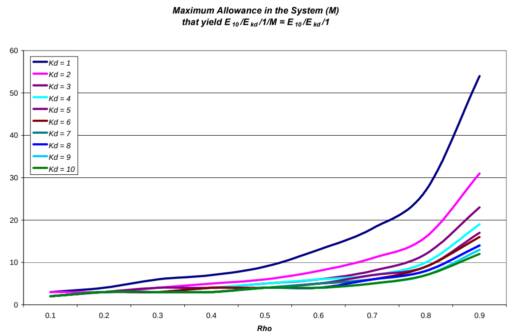

Figure 8: The non decreasing function of M for given K_a of 10 and K_d from 1 to 10

from the illustrated Figure 7 and Figure 8, we can see that the maximum number in the system (M) has exhibited the non decreasing exponential like function relative to Rho (ρ), but rather decreasing as K_a increase, from K_a = 1 and K_d = 10 that required M = 54 in Figure 7 to K_a = 10 and K_d = 10 that required M = 12 in Figure 8 for a given ρ = 0.9. This is shown that as the occupancy rate (ρ) increase, the system required more queue line in exponentially increasing rate and we can further estimate that as we increase the stages of arrival (K_a) and stages of departure (K_d), or as the statistical distribution of workers/researchers' interest that differ from traditional exponential distribution in the lower Cov ranges, the system requires less capacity (M, imply less queue line) in order to achieved the probability of full system (P_M) of 0.00.

The Predictable Pattern of the System Measures

As the results of all system measures ranging from $E_1/E_1/1$ to $E_{10}/E_{10}/1$ form PSA and $E_{11}/E_{11}/1$ to $E_\infty/E_\infty/1$ from simulation (in order to minimize the calculation duration, we use 10,000 samples), coefficient of variation (Cov_{Ta} and Cov_{Ts}) of 1.00 to 0.00, we found the interesting pattern as some samples illustrated below:

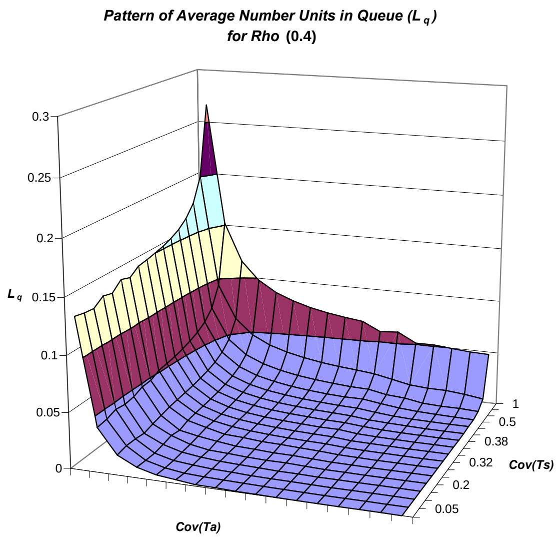

Figure 9: Pattern of Average Number of Units in Queue (L_q) with occupancy rate (ρ) of 0.4

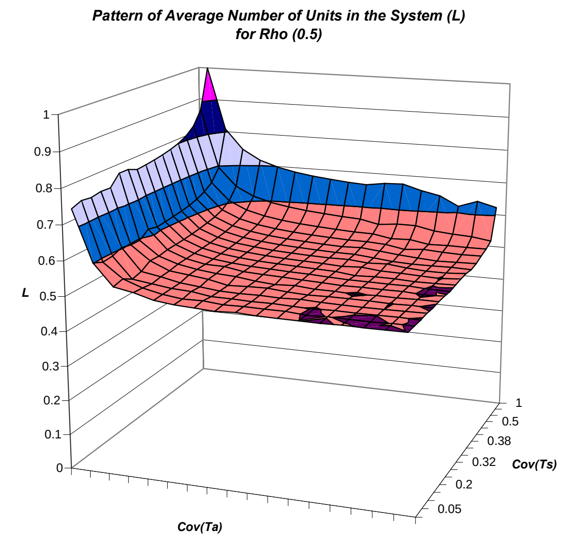

Figure 10: Pattern of Average Number of Units in the System (L) with occupancy rate (ρ) of 0.5

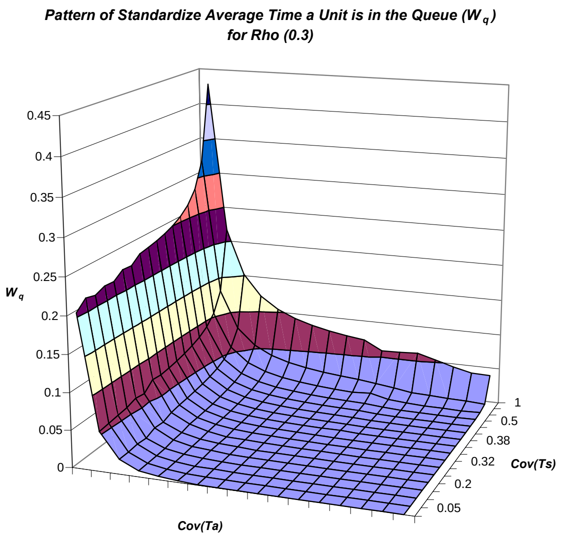

Figure 11: Pattern of Standardize Average Time a Unit is in the Queue (W_q) with occupancy rate (ρ) of 0.3

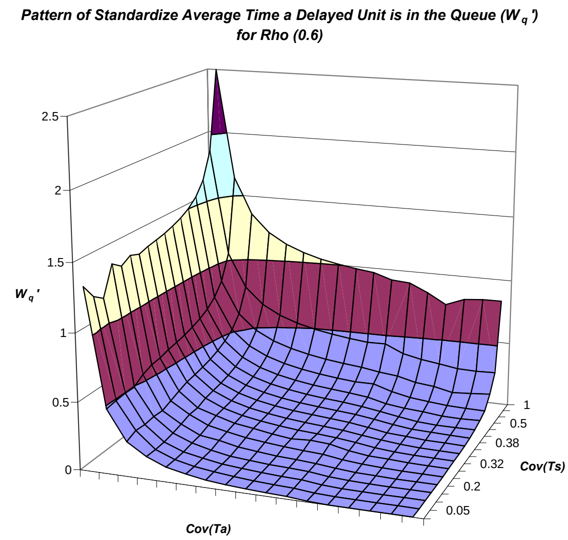

Figure 12: Pattern of Standardize Average Time a Delayed Unit is in the Queue (W_q') with occupancy rate (ρ) of 0.6

from the above figures, we can clearly see the predictable pattern of the system measures from the distribution with extreme low coefficient of variation (Cov_{Ta}, Cov_{Ts} < 0.32) or extreme high independent stages of arrival and departure ($10 < K_a, K_d < \infty$) that shown to approach a certain value, interpretation will allow estimating the statistic measures for any *Cov* from 0.00 to 1.00.

CONCLUSION AND REMARKS

This paper shows how to calculate standardize system statistics for the Erlang distributed interarrival and service time, determined by low coefficient of variation, to facilitate the arbitrary interarrival and arbitrary service time. In order to yield more precise queueing statistics given by any statistic distribution that differs from traditional exponential queueing model.

Plumchitchom invented and illustrated the *Possible Probability Stages Searching Algorithm* (PSA), proven to be the most effective and accurate way, to construct equilibrium equations in the complex P_{nij}s environments, to further derive system measures for interarrival and service time distribution with extreme low coefficient of variation.

Table 1 to Table 6 provide system measures that result from PSA and Simulation with 99.9995% and 99% accuracy, respectively.

The PSA shown in this paper is the introduction to Erlang queueing system and can only derive system measures for one service facility. Workers/researchers, who have the comprehensive understanding of the algorithm, therefore can easily make adjustment for desired number of service facilities, tandem queueing system, or multiple arrival and/or multiple service stages.

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